

Marginal deformations of $\mathcal{N} = 4$ SYM and open vs. closed string parameters

Manuela Kulaxizi

C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794-3840, USA

Received 4 July 2014; received in revised form 11 August 2014; accepted 12 August 2014

Available online 19 August 2014

Editor: Herman Verlinde

Abstract

We make precise the connection between the generic Leigh–Strassler deformation of $\mathcal{N} = 4$ SYM and noncommutativity. We construct an appropriate noncommutativity matrix, which turns out to define a nonassociative deformation. Viewing this noncommutativity matrix as part of the set of open string data which characterize the deformation and mapping them to the closed string data (e.g. metric and B-field), we are able to construct the gravity dual and the corresponding deformed flat space geometry up to third order in the deformation parameter ρ .

Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/3.0/>). Funded by SCOAP³.

1. Introduction

The AdS/CFT correspondence [72,44,108] offers an equivalence between gauge theory and gravity. In its original form, it relates superconformal $\mathcal{N} = 4$ $SU(N)$ Super Yang–Mills to closed string theory on $AdS_5 \times S^5$ with N units of RR-flux. While closed strings on nontrivial backgrounds with RR-fluxes are still in many ways intractable, their low energy description in terms of supergravity is not. From the gauge theory point of view, this limit corresponds to large N and strong 't Hooft coupling λ . This makes the correspondence extremely useful in that it provides a window into understanding the physics of gauge theories in a region that is otherwise difficult to explore. By now the original proposal has been greatly extended covering gauge theories with less amount of supersymmetry and/or a running coupling constant [59,60,74,57,66,38].

The simplest extensions of the original AdS/CFT proposal arise by considering supersymmetry preserving deformations of the $\mathcal{N} = 4$ SYM theory; exactly marginal and/or relevant

deformations. Naively, the gravity dual backgrounds of the former would be more accessible than those of the latter. It turns out however that the opposite is true. In fact, the gravity duals of a class of supersymmetric mass deformations were discovered quite early on; see for example [90,91,48] and references therein. The main reason is that these backgrounds can be analyzed using the truncation to five-dimensional supergravity, something which is not possible for marginal deformations of the $\mathcal{N} = 4$ theory. Actually it was only fairly recently that the authors of [69] succeeded in constructing the corresponding backgrounds for a subclass of these latter theories.¹

Marginal deformations of $\mathcal{N} = 4$ SYM preserve $\mathcal{N} = 1$ supersymmetry and are mainly described by two parameters, denoted as β and ρ , in addition to the gauge coupling g_{YM} . In [69] Lunin and Maldacena discovered the geometry dual to the β -deformed theory, i.e. when $\rho = 0$. In this case apart from the $U(1)$ R symmetry, the theory preserves additional two $U(1)$ global symmetries. These symmetries played a significant part in the construction of the new solution. When $\rho \neq 0$, however, the theory does not preserve any continuous symmetries other than the $U(1)$ R-symmetry (only a discrete $Z_3 \times Z_3$ symmetry) and the problem of finding the dual gravitational background has resisted solution.

In this note, we revisit the question of how to obtain the gravitational dual of the ρ -deformation. The starting point is to make precise the description of the deformation in the dual gravity theory as a non-commutative deformation of the transverse space. The relation between exactly marginal deformations of $\mathcal{N} = 4$ SYM and non-commutativity was actually established early on [11] (see also [64,63] for a non-commutative description at weak coupling). For the β -deformation it was only made explicit recently, in [69]. Here we attempt to make the relation to non-commutativity explicit for the ρ -deformation as well. In particular, we construct the non-commutativity matrix Θ which practically realizes the deformation. The $(2, 0)$ and $(0, 2)$ parts of the non-commutativity matrix are easily obtained from the F-term constraints. To specify the $(1, 1)$ components, we follow the discussion in [62]. We first consider all possibilities allowed by the global discrete $Z_2 \times Z_3$ symmetries. However, symmetries do not adequately constrain the form of Θ . To determine it completely we transform it to spherical coordinates and require that it be real, transverse and independent of the radial direction. These restrictions are imposed upon us from the exact marginality of the deformation and specify Θ completely.

It turns out that we can perform a rather non-trivial check on the proposed non-commutativity matrix. There exist points in the Leigh–Strassler deformation space parametrized by (β, ρ) , which are related to each other by a field redefinition. For instance, the $\mathcal{N} = 4$ SYM theory deformed by $(\beta_1 = 0, \rho_1 \in R)$ is equivalent to the same theory deformed by (β_2, ρ_2) such that $\rho_2 = \frac{i\rho_1}{\sqrt{3}} = \frac{i\beta_2}{2}$. Clearly, the associated non-commutativity matrices Θ_1, Θ_2 should also be related if they correctly describe the deformations. We find that this is indeed the case; a simple coordinate transformation takes us from Θ_1 to Θ_2 confirming the equivalence of the two theories in this description. A disconcerting fact about Θ is that it does not satisfy the associativity condition. Hence, we do not have a star-product formulation which would enable us to express the superpotential of the deformed theory in terms of the parent $\mathcal{N} = 4$ theory (this was successfully done for the case of the β -deformation [69] — see also [58]).

One may wonder whether we can use the knowledge of the non-commutativity matrix to obtain information on the gravity dual of the theory. A way to address this question, is perhaps to consider the non-commutativity matrix as part of the gauge theory or open string theory data

¹ For a general approach on how to find the gravity dual a description of a given gauge theory see for instance [103, 102,38–41,107] and references therein.

(metric + non-commutativity) which characterize the deformed theory. One might then hope to obtain some of the closed string data (metric + B-field) using the relations of Seiberg and Witten [98] (see also [96]).

We examine this possibility by directly employing the Seiberg–Witten relations for the ρ -deformation. Unfortunately, the non-commutativity matrix which describes the ρ -deformation does not satisfy the associativity condition. Most likely the Seiberg–Witten relations are not valid for non-associative deformation parameters. Nevertheless, nonassociativity is a second-order in ρ effect, so one may hope to at least obtain a correct result up to third order in the deformation parameter. Indeed we find that the set of closed string data deduced from the Seiberg–Witten relations, i.e. metric, dilaton and B-field, satisfy the field equations of supergravity up to third order in the deformation parameter ρ . Since all the necessary symmetries are built in, we expect that this deformed flat space geometry presents, up to third order in the deformation parameter, the background where once D-branes are immersed and the near horizon limit is taken, the AdS dual geometry will be recovered.

Given this result, we consider the effective action of the ρ -deformed gauge theory, obtained by giving a vacuum expectation value to one of the scalars and integrating out the massive fields. According to [73,71,106,80,22,8], the leading IR large N part of this action should coincide with the DBI action for a D3-brane immersed in the dual background. We observe that in the case of the β -deformed gauge theory, the corresponding DBI action is characterized by the open string data $(\mathcal{G}_{\text{AdS}_5 \times S^5}, \Theta)$ and that the associated NS–NS closed string fields (g, B) are part of the *exact* Lunin–Maldacena solution. This is not surprising. Indeed, the Lagrangian description of this theory can be given in terms of the $\mathcal{N} = 4$ Lagrangian with the product of matter fields replaced by a star product of the Moyal type. Subsequently, all amplitudes in the planar limit can be shown [58] to be proportional up to a phase to their $\mathcal{N} = 4$ counterparts. Then the open string data $(\mathcal{G}_{\text{AdS}_5 \times S^5}, \Theta = 0)$ of the $\mathcal{N} = 4$ SYM theory are naturally promoted to the set $(\mathcal{G}_{\text{AdS}_5 \times S^5}, \Theta)$. Can something similar occur for the ρ -deformation?

Non-associativity again creates a potential problem: planar equivalence with the parent $\mathcal{N} = 4$ theory using a star-product is far from obvious. Nonetheless, nonassociativity is a second-order in ρ effect, so we can safely assume that $(\mathcal{G}_{\text{AdS}_5 \times S^5}, \Theta)$ describe the deformation up to this order in the deformation parameter. We then map the open string fields to the closed ones using the Seiberg–Witten relations and obtain the gravity dual of the ρ -deformed gauge theory up to third order in ρ .

The structure of this paper is the following. In Section 2 we review some known facts about marginal deformations of the $\mathcal{N} = 4$ theory and their gravity duals. In addition, we explore some special points in the deformation space for which the general theory with $\beta \neq 0$ and $\rho \neq 0$ is equivalent to an exactly marginal deformation with either $\tilde{\beta} = 0$ or $\tilde{\rho} = 0$. In Section 3, we use the logic outline in [62] and determine the noncommutativity matrix for the ρ -deformation. Viewing Θ as part of the open string data which describe the deformation, we use the Seiberg–Witten relations to find the corresponding closed string data (g, B) . This procedure is illustrated in Section 4 where we derive the ρ -deformed flat space geometry up to third order in the deformation parameter. In Section 5 we proceed with considerations on the DBI action which provide us with the gravity dual of the ρ -deformed theory to the same order. We conclude in Section 6.

2. The Leigh–Strassler deformation

Not long after it was realized that $\mathcal{N} = 4$ $SU(N)$ Super Yang–Mills theory is finite (see e.g. [104] for an account), it became clear that it might not be the only four dimensional theory with

that property [55,87,56,88,43]. It was however almost ten years later, when Leigh and Strassler undertook a systematic study of marginal deformations of $\mathcal{N} = 4$ and indeed showed that there exists a whole class of $\mathcal{N} = 1$ supersymmetric gauge theories satisfying both the requirements for conformal invariance and finiteness [67]. More precisely, they showed that the $\mathcal{N} = 4$ theory admits a three-complex-parameter family of marginal deformations which preserve $\mathcal{N} = 1$ supersymmetry and are described by the following superpotential:

$$\mathcal{W} = i h \text{Tr}[(e^{i\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\beta} \Phi_1 \Phi_3 \Phi_2) + \rho(\Phi_1^3 + \Phi_2^3 + \Phi_3^3)] \quad (1)$$

where Φ^I with $I = 1, 2, 3$ are the three chiral superfields of the theory. Together with the gauge coupling g_{YM} , the complex parameters (h, β, ρ) that appear in the superpotential constitute the four couplings of the theory.

While it is clear at the classical level that these deformations are marginal — since all operators of the component Lagrangian have classical mass dimension equal to four — this is not necessarily true quantum mechanically. Leigh and Strassler realized that by using the constraints of $\mathcal{N} = 1$ supersymmetry and the exact NSVZ beta-functions [85,100,101] written in terms of the various anomalous dimensions of the theory, it was possible to express the conditions for conformal invariance of the quantum theory, through linearly dependent equations which were therefore likely to have nontrivial solutions. In this way, they were able to demonstrate that the deformation of (1) is truly marginal at the quantum level, so long as the four couplings of the theory satisfy a single complex constraint $\gamma(g_{\text{YM}}, h, \beta, \rho) = 0$. In other words, there exists a three-complex-dimensional surface $\gamma(g_{\text{YM}}, h, \beta, \rho) = 0$ in the space of couplings, where both beta functions and anomalous dimensions vanish and thus the $\mathcal{N} = 1$ gauge theories mentioned above are indeed conformally invariant. In general, the function γ is not known beyond two-loops [3,93,84,33,78] in perturbation theory, where it reads:

$$|h|^2 \left[\frac{1}{2} \left(|q|^2 + \frac{1}{|q|^2} \right) - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 + |\rho|^2 \left(\frac{N^2 - 4}{2N^2} \right) \right] = g_{\text{YM}}^2 \quad (2)$$

with q defined as $q = e^{i\beta}$ and N the number of colours of the gauge theory.

For the β -deformed gauge theory, i.e., obtained by setting $\rho = 0$ in the superpotential of Eq. (1), the Leigh–Strassler constraint at two loops can be written as:

$$|h|^2 \left[\frac{1}{2} \left(|q|^2 + \frac{1}{|q|^2} \right) - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 \right] = g_{\text{YM}}^2 \quad (3)$$

In this case, one immediately notices that when $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ therefore $|q| = 1$, (3) reduces to:

$$|h|^2 \left[1 - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 \right] = |h|^2 \left(1 - \frac{4}{N^2} \sin^2 \beta_{\mathbb{R}} \right) = g_{\text{YM}}^2 \quad (4)$$

which in the large N limit yields: $|h|^2 = g_{\text{YM}}^2$. Despite the fact that this result was obtained from the two-loop expression of the conformal invariance condition, it has been shown to be true to all orders in perturbation theory at the planar limit [58] (see also [65,78]). Actually the author of [58] went even further and showed that all planar amplitudes in the $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ theory are proportional to their $\mathcal{N} = 4$ counterparts, thus explicitly proving finiteness and conformal invariance. The proof made use of an existing proposal [69] for an equivalent “noncommutative” realization of the theory. For the more general case of complex $\beta = \beta_{\mathbb{R}} + i\beta_{\mathbb{I}}$, Eq. (3) in the planar limit reads:

$$\frac{1}{2}|h|^2\left(|q|^2 + \frac{1}{|q|^2}\right) = |h|^2 \cosh(2\beta_{\mathbb{I}}) = g_{\text{YM}}^2 \quad (5)$$

It is then evident that the coupling constant h receives corrections with respect to its $\mathcal{N} = 4$ SYM value. Nevertheless, diagrammatic analysis [58] showed that all planar amplitudes with external gluons are equal to those of the $\mathcal{N} = 4$ theory up to a five-loop level. To this order and beyond, it is most likely that the planar equivalence between the parent theory and its deformation will break down. For (more) recent investigation on β -deformations from the gauge theory point of view see [33,78,94,95,89,77,32].

Special points along the deformation occur when $q = e^{i\beta}$ is a root of unity. These points have been studied early on [12,11] with a dual interpretation as orbifolds with discrete torsion. The marginally deformed theories have been further explored in [28,27,26], and several remarkable properties have been demonstrated. In particular, it was shown that as expected, the S-duality of $\mathcal{N} = 4$ extends to their space of vacua, and that, again for special values of β , there are also new Higgs branches on moduli space. These are mapped by S-duality to completely new, confining branches which appear only at the quantum level. Furthermore, at large N the Higgs and confining branches can be argued to be described by Little String Theory [26]. Finally, the possibility of an underlying integrable structure for the deformed theories in analogy with $\mathcal{N} = 4$ SYM, was investigated at special values of the deformation parameter in [10,17] and for generic β in [9,34,79,6].

2.1. Marginal deformations and gauge/gravity duality

A natural place to explore theories that arise as marginal deformations of $\mathcal{N} = 4$ $SU(N)$ SYM is the AdS/CFT correspondence where the strong coupling regime of the undeformed theory is realized as weakly coupled supergravity on $\text{AdS}_5 \times S^5$. Due to superconformal symmetry, the dual gravitational description of these theories is expected to be of the form: $\text{AdS}_5 \times \tilde{S}^5$ with \tilde{S}^5 a sphere deformed by the presence of additional NS–NS and RR fluxes. Indeed in [2], where the dual background was constructed to second order in the deformation parameters, it was shown that apart from the already present five-form flux one should also turn on (complexified) three-form flux $G_{(3)}$ along the S^5 .

Essential progress however in this direction was only recently achieved through the work of Lunin and Maldacena [69]. The authors of [69] succeeded in finding the *exact* gravity dual of the β -deformed gauge theory.

In this case, apart from the $U(1)_R$ R-symmetry the theory preserves two global $U(1)$ s, which act on the superfields in the following way:

$$\begin{aligned} U(1)_1 : \quad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\alpha_1} \Phi_2, e^{-i\alpha_1} \Phi_3) \\ U(1)_2 : \quad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow (e^{-i\alpha_2} \Phi_1, e^{i\alpha_2} \Phi_2, \Phi_3) \end{aligned} \quad (6)$$

The main idea underlying the solution generating technique proposed in [69], was the natural expectation that the two $U(1)$ symmetries preserved by the deformation would be realized geometrically in the dual gravity solution. For $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ their prescription amounts to performing an $SL(2, \mathbb{R})$ transformation on the complexified Kähler modulus τ of the two torus associated with the $U(1)$ symmetries in question. The specific element of $SL(2, \mathbb{R})$ under consideration is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. It is chosen so as to ensure that the new solution will present no singularities as long as the original one is non-singular and its sole free parameter c is naturally identified with the real deformation parameter $\beta_{\mathbb{R}}$ of the gauge theory.

Later on, the method of Lunin and Maldacena was reformulated [19] in terms of the action of a T-duality group element on the background matrix $E = g + B$ providing a significantly easier way of obtaining the new solutions. In particular, it was shown [19] that one can embed the $SL(2, \mathbb{R})$ that acts on the Kähler modulus into the T-duality group $O(3, 3, \mathbb{R})$ in the following way:

$$\mathcal{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{\Gamma} & \mathbf{1} \end{pmatrix} \quad \text{where now } \mathbf{\Gamma} \equiv \begin{pmatrix} 0 & -\beta_{\mathbb{R}} & \beta_{\mathbb{R}} \\ \beta_{\mathbb{R}} & 0 & -\beta_{\mathbb{R}} \\ -\beta_{\mathbb{R}} & \beta_{\mathbb{R}} & 0 \end{pmatrix} \quad (7)$$

where $\mathbf{1}$ and $\mathbf{0}$ represent the 3×3 identity and zero matrices respectively. Suppose then that $E_0 = g_0 + B_0$ denotes the part of the original supergravity background along the $U(1)$ isometry directions which are to be deformed. Acting on E_0 with the T-duality group element \mathcal{T} of (7) one obtains the NS–NS fields of the deformed solution in terms of E_0 and $\mathbf{\Gamma}$ according to:

$$E = \frac{1}{E_0^{-1} + \mathbf{\Gamma}} \\ e^{2\Phi} = e^{2\Phi_0} \det(1 + E_0 \mathbf{\Gamma}) \equiv e^{2\Phi_0} G \quad (8)$$

The RR-fields of the background can be computed using the T-duality transformation rules of [105,13,24,37,50], however the details of this transformation need not concern us here. As an example, let us consider ten-dimensional flat space parametrized as:

$$ds^2 = -dt^2 + \sum_{\mu=1}^3 dx^\mu dx_\mu + \sum_{i=1}^3 (dr_i^2 + r_i^2 d\varphi_i^2) \quad (9)$$

In this case E_0 will contain the components of the flat metric along the polar angles φ_i . Applying Eqs. (8) yields:

$$ds^2 = -dt^2 + \sum_{\mu=1}^3 dx^\mu dx_\mu + \sum_{i=1}^3 (dr_i^2 + Gr_i^2 d\varphi_i^2) + \beta_{\mathbb{R}} Gr_1^2 r_2^2 r_3^2 \left(\sum_{i=1}^3 d\varphi_i \right)^2 \\ e^{2\Phi} = G, \quad G^{-1} = 1 + \beta_{\mathbb{R}}^2 \left(\sum_{i \neq j} r_i^2 r_j^2 \right), \quad B = \beta_{\mathbb{R}} G \left(\sum_{i \neq j} r_i^2 r_j^2 d\varphi_i d\varphi_j \right) \quad (10)$$

This is the deformed flat space geometry where by placing D3-branes at the origin and taking the near horizon limit, one obtains the gravity dual to the β -deformed gauge theory. Alternatively, the latter background can be constructed by applying (8) on $\text{AdS}_5 \times S^5$ representing the dual gravitational description of the undeformed parent $\mathcal{N} = 4$ theory:

$$ds^2 = R^2 (ds_{\text{AdS}_5}^2 + ds_5^2), \quad \text{where: } ds_5^2 = \sum_i (d\mu_i^2 + G\mu_i^2 d\varphi_i^2) + \hat{\beta} G \mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_i d\varphi_i \right)^2 \\ e^{2\Phi} = e^{2\Phi_0} G, \quad G^{-1} = 1 + \hat{\beta}^2 \left(\sum_{i \neq j} \mu_i^2 \mu_j^2 \right), \quad \hat{\beta} = R^2 \beta_{\mathbb{R}}, \quad R^4 = 4\pi e^{\Phi_0} N \\ B = \hat{\beta} R^2 G \left(\sum_{i \neq j} \mu_i^2 \mu_j^2 d\varphi_i d\varphi_j \right), \quad C_2 = -\beta_{\mathbb{R}} (16\pi N) \omega_1 \left(\sum_i d\varphi_i \right) \\ F_5 = (16\pi N) (\omega_{\text{AdS}_5} + G\omega_{S^5}), \quad \omega_{S^5} = d\omega_1 d\varphi_1 d\varphi_2 d\varphi_3, \quad \omega_{\text{AdS}_5} = d\omega_4 \quad (11)$$

Reformulating the Lunin–Maldacena generating solution technique in terms of the T-duality group action, made especially transparent its relation to similar methods employed in the context of noncommutative gauge theories.²

It is easy to see that Γ of (7) is precisely the noncommutativity matrix Θ associated with the deformation of the transverse space. In [62] the possibility of determining Θ directly from the gauge theory Lagrangian (and some basic notions of AdS/CFT) was discussed. In particular, it was shown that by promoting the matter fields to coordinates $(z^I, \bar{z}^{\bar{I}})$ and requiring that Θ should be real, preserve the global symmetries of the theory and respect exact marginality,³ Θ was uniquely fixed to be:

$$\Theta_\beta = a \begin{pmatrix} 0 & z_1 z_2 & -z_1 z_3 & 0 & -z_1 \bar{z}_2 & z_1 \bar{z}_3 \\ -z_1 z_2 & 0 & z_2 z_3 & \bar{z}_1 z_2 & 0 & -z_2 \bar{z}_3 \\ z_3 z_1 & -z_2 z_3 & 0 & -\bar{z}_1 z_3 & \bar{z}_2 z_3 & 0 \\ 0 & -\bar{z}_1 z_2 & \bar{z}_1 z_3 & 0 & \bar{z}_1 \bar{z}_2 & -\bar{z}_1 \bar{z}_3 \\ z_1 \bar{z}_2 & 0 & -\bar{z}_2 z_3 & -\bar{z}_1 \bar{z}_2 & 0 & \bar{z}_2 \bar{z}_3 \\ -\bar{z}_3 z_1 & \bar{z}_3 z_2 & 0 & \bar{z}_1 \bar{z}_3 & -\bar{z}_2 \bar{z}_3 & 0 \end{pmatrix} \quad (12)$$

with $a = 2 \sin \beta_{\mathbb{R}}$. Θ_β appears to be different from Γ of (7), but transforming it to polar coordinates (r_i, φ_i) on \mathbb{R}^6 one finds that

$$\Theta_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ 0 & 0 & 0 & a & 0 & -a \\ 0 & 0 & 0 & -a & a & 0 \end{pmatrix} \quad (13)$$

thereby showing that Θ_β and Γ are effectively the same (recall that the Lunin–Maldacena solution is valid for small $\beta_{\mathbb{R}}$ in which case $a \simeq 2\beta_{\mathbb{R}}$).

To obtain the dual background for the general case of complex β one needs to perform an additional $SL(2, \mathbb{R})_s$ transformation on the solution corresponding to $\beta_{\mathbb{R}}$. By $SL(2, \mathbb{R})_s$ we denote here the $SL(2, \mathbb{R})$ symmetry of ten dimensional type IIB supergravity which acts nontrivially on the complexified scalar and two-form fields of the theory. Being a symmetry of the equations of motion it can be used to generate distinct solutions. Subsequent work on the subject of the β -deformed gauge theories has provided further checks of the AdS/CFT correspondence [36,35, 21,29,42,53] whereas generalizations as well as applications of the solution generating technique introduced in [69] were considered in [45,34,53,4,5,92].

2.2. Special points along the general Leigh–Strassler deformation

In this article we will be mainly interested in the ρ -deformed gauge theories. In this case — when $\rho \neq 0$ — the theory does not preserve additional $U(1)$ symmetries, it is however invariant under a global discrete symmetry $\mathbb{Z}_3 \times \mathbb{Z}_3$ acting on the superfields as:

$$\begin{aligned} \mathbb{Z}_{3(1)} : \quad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_3, \Phi_1, \Phi_2) \\ \mathbb{Z}_{3(2)} : \quad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow \left(\Phi_1, e^{\frac{i2\pi}{3}} \Phi_2, e^{-\frac{i2\pi}{3}} \Phi_3 \right) \end{aligned} \quad (14)$$

² Evidence relating marginal deformations and noncommutativity was given earlier both at strong [11] and weak [64, 63] coupling.

³ More details on this last requirement will be given in Section 3.

As previously mentioned, the presence of global $U(1)$ s is crucial in the solution generating technique of Lunin and Maldacena which is therefore not applicable here. In fact, the exact gravity dual for this case is still unknown. Despite however that the absence of extra continuous symmetries makes the cases of $\rho = 0$ and $\rho \neq 0$ radically different, there exist special points along the space of couplings where the two theories are not only similar but actually equivalent.

As first pointed out in [11] — see also [17,16] — it is possible to start with either the set $(\beta, \rho) = (\beta, 0)$ or $(\beta, \rho) = (0, \rho)$, and via a field redefinition reach a point in the deformation space with $(\tilde{\beta} \neq 0, \tilde{\rho} \neq 0)$. The final point will obviously not represent the most general deformation, since the new couplings $\tilde{\beta}$ and $\tilde{\rho}$ will be given in terms of the original parameter. In other words, there will exist a function $f(\tilde{\beta}, \tilde{\rho}) = 0$ relating the two. Furthermore, requiring that the field redefinition be the result of a unitary transformation imposes a restriction on the original value of the coupling; be it β or ρ . In particular, suppose that we consider the marginally deformed theory at the point $(\beta, \rho = 0)$ and then take:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \rightarrow \begin{pmatrix} A & A & A \\ B & \omega B & \omega^2 B \\ C & \omega^2 C & \omega C \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \quad (15)$$

with Φ_I the three chiral superfields and $\omega = e^{i2\pi/3}$ the third root of unity. Note here that since the deformation enters only in the superpotential, it suffices to consider transformations that affect the chiral fields independently from the antichiral ones. In other words, we do not expect mixing between holomorphic and antiholomorphic pieces. If we furthermore impose the following conditions on the free parameters A, B, C : $|A| = |B| = |C| = \frac{1}{\sqrt{3}}$ and $ABC = \pm \frac{i\lambda}{3\sqrt{1+2\cos 2\beta}}$ with $\lambda \in \mathbb{C}$, we find that the original β -deformed gauge theory is equivalent to the marginally deformed $\mathcal{N} = 4$ SYM theory with coupling constants:

$$\tilde{\rho} = \pm \frac{2 \sin \beta}{3\sqrt{1+2\cos 2\beta}} \quad \text{and} \quad e^{i\tilde{\beta}} = \pm \frac{2 \cos(\beta - \frac{\pi}{6})}{\sqrt{1+2\cos 2\beta}} \quad (16)$$

provided that $\beta = \beta_{\mathbb{R}} + i\beta_{\mathbb{I}}$ satisfies the following equation:

$$4 \cos 2\beta_{\mathbb{R}} \cos 2\beta_{\mathbb{I}} + 4 \cos^2 2\beta_{\mathbb{R}} + 4 \cos^2 2\beta_{\mathbb{I}} - 3(1 + 3\lambda) = 0 \quad (17)$$

Solutions to (17) define special regions in the coupling constant space where the Leigh–Strassler theory with generic β and $\rho = 0$ is equivalent to a theory with both $\tilde{\beta}$ and $\tilde{\rho}$ nonvanishing but constrained to satisfy a specific relation dictated from (16). It is worth remarking here that there is no solution of (16) and (17) for which *both* β and $\tilde{\beta}$ are real. This is particularly interesting, because it is only for the β -deformed gauge theory with $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ that a precise connection with noncommutativity is possible. It is natural to wonder whether distinct unitary field redefinitions of a type similar to (15) could take us from different β 's to different $\tilde{\beta}$ and $\tilde{\rho}$. It is however not hard to deduce that up to a phase in $\tilde{\rho}$ — which can be reabsorbed in the definition of the coupling constant h — and a sign in $\tilde{\beta}$, all such unitary transformations share the same starting point (17) and lead to the same theory (16).

In an analogous manner one can find specific values of ρ for which the theory with $\beta = 0$ is equivalent to another one with both couplings $\tilde{\beta}$ and $\tilde{\rho}$ turned on. Detailed analysis in this case shows in fact that such a mapping is possible for *any* original value of ρ with parameters $\tilde{\rho}$ and $\tilde{\beta}$ given by:

$$\tilde{\rho}^2 = -\frac{\rho^2}{\rho^2 + 3}, \quad \text{and} \quad \sin^2 \tilde{\beta} = -\tilde{\rho}^2 = \frac{\rho^2}{\rho^2 + 3} \quad (18)$$

The precise field redefinition through which this is achieved, is of the form of (15):

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \quad (19)$$

Note here again that $\tilde{\beta} = \tilde{\beta}_{\mathbb{R}} \in \mathbb{R}$ if and only if $\rho \in \mathbb{R}$ which implies that $\tilde{\rho} \in \mathbb{I}$. If one additionally assumes that $\tilde{\beta}_{\mathbb{R}} \in \mathbb{R} \ll 1$ then the deformed theory with $\beta = 0$ and $\rho = q_1 \in \mathbb{R}$ is equivalent to a theory with $2 \sin \tilde{\beta} = \pm 2 \frac{q_1}{\sqrt{3}}$ and $\tilde{\rho} = \pm i \frac{q_1}{\sqrt{3}} \in \mathbb{I}$. In Section 3, we will see that this particular point in the deformation space naturally shows up in the noncommutative description of the moduli space. This will provide us with a non-trivial check on the consistency of the noncommutative interpretation.

So far we have looked at special points in the space of couplings which can be studied at the level of the gauge theory Lagrangian. There are however a couple of interesting observations one can additionally make on the basis of the Leigh–Strassler constraint as this is given in Eq. (2). Notice first that (2) reduces in the planar limit to:

$$|h|^2 \left[\frac{1}{2} \left(|q|^2 + \frac{1}{|q|^2} \right) + \frac{1}{2} |\rho|^2 \right] = |h|^2 \left[\cosh(2\beta_{\mathbb{I}}) + \frac{1}{2} |\rho|^2 \right] = g_{\text{YM}}^2 \quad (20)$$

This implies that when $\rho \neq 0$ the coupling constant h at the conformal fixed point will be different from g_{YM} , in contrast to what happens for $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$. In this sense, turning on ρ is similar to turning on the imaginary part of $\beta = \beta_{\mathbb{I}}$. Yet, there seems to exist a particular point in the deformation space for which $h = g_{\text{YM}}$ continues to hold in the large N limit. This occurs when:

$$\cosh(2\beta_{\mathbb{I}}) + \frac{1}{2} |\rho|^2 = 1 \quad \Rightarrow \quad \beta_{\mathbb{I}} = \frac{1}{2} \arg \cosh \left(1 - \frac{|\rho|^2}{2} \right) \quad (21)$$

Closer inspection however of (21) reveals that it has no possible solutions, assuming $\beta_{\mathbb{I}} \in \mathbb{R}$ and $|\rho| > 0$. This implies that despite appearances, there is no special point for which $h = g_{\text{YM}}$ at two loops in the planar limit. Naturally, one expects that an analogous equation relating the two couplings, for which $h = g_{\text{YM}}$ at large N , may arise at any order in perturbation theory. What is not clear of course, is whether it will generically have any solutions or not.

3. Marginal deformations and noncommutativity

In [62] we showed that for the β -deformed gauge theory it is possible to construct a noncommutativity matrix Θ encoding in a precise manner information on the moduli space of the theory. This construction is very simple and is based on fundamental properties of the gauge theory and AdS/CFT. In what follows we will adopt the reasoning outlined in [62] in order to determine a noncommutativity matrix for the ρ -deformation. We set $\beta = 0$ for the time being and later on discuss how to incorporate $\beta \neq 0$.

Our starting point is the F-term constraints:

$$\begin{aligned} \Phi_1 \Phi_2 &= \Phi_2 \Phi_1 + \rho \Phi_3^2, & \Phi_2 \Phi_3 &= \Phi_3 \Phi_2 + \rho \Phi_1^2, & \Phi_3 \Phi_1 &= \Phi_1 \Phi_3 + \rho \Phi_2^2 \\ \bar{\Phi}_1 \bar{\Phi}_2 &= \bar{\Phi}_2 \bar{\Phi}_1 - \bar{\rho} \bar{\Phi}_3^2, & \bar{\Phi}_2 \bar{\Phi}_3 &= \bar{\Phi}_3 \bar{\Phi}_2 - \bar{\rho} \bar{\Phi}_1^2, & \bar{\Phi}_3 \bar{\Phi}_1 &= \bar{\Phi}_1 \bar{\Phi}_3 - \bar{\rho} \bar{\Phi}_2^2 \end{aligned} \quad (22)$$

from which we read the holomorphic and antiholomorphic parts of Θ interpreting the eigenvalues of these matrices in the large N limit, as noncommuting coordinates parametrizing the space transverse to the worldvolume of the D3-brane. More precisely we have:

$$\begin{aligned} [z_1, z_2] &= \rho z_3^2, & [z_2, z_3] &= \rho z_1^2, & [z_3, z_1] &= \rho z_2^2 \\ [\bar{z}_1, \bar{z}_2] &= -\bar{\rho} \bar{z}_3^2, & [\bar{z}_2, \bar{z}_3] &= -\bar{\rho} \bar{z}_1^2, & [\bar{z}_3, \bar{z}_1] &= -\bar{\rho} \bar{z}_2^2 \end{aligned} \quad (23)$$

Following [62] we would like to assume that there exists a star product between some *commuting* variables $z^I, \bar{z}^{\bar{I}}$ which leads to commutation relations analogous to (23), so that we can write for instance: $i\Theta^{1\bar{2}} = [z_1, z_2]_* = z_1 * z_2 - z_2 * z_1 = \rho z_3^2$. This enables us to define a noncommutativity matrix which although position dependent, its entries are ordinary commuting objects. Then, under a change of coordinates Θ will transform as a contravariant antisymmetric tensor field. We therefore write Θ in matrix form as:

$$\Theta = \begin{pmatrix} 0 & i\rho z_3^2 & -i\rho z_2^2 & & & \\ -i\rho z_3^2 & 0 & i\rho z_1^2 & & ? & \\ i\rho z_2^2 & -i\rho z_1^2 & 0 & & & \\ & & & 0 & -i\bar{\rho} \bar{z}_3^2 & i\bar{\rho} \bar{z}_2^2 \\ & ? & & i\bar{\rho} \bar{z}_3^2 & 0 & -i\bar{\rho} \bar{z}_1^2 \\ & & & -i\bar{\rho} \bar{z}_2^2 & i\bar{\rho} \bar{z}_1^2 & 0 \end{pmatrix} \quad (24)$$

It is clear that the F-term constraints determine the (2, 0) and (0, 2) parts of Θ . D-terms will in principle specify the (1, 1) pieces of the noncommutativity matrix. However, as demonstrated in [62], there is an alternative indirect way of acquiring the information pertaining to D-terms. Recall that for the β -deformed gauge theory it was possible to fully determine Θ by imposing certain simple conditions on its form — namely definite reality properties, symmetries and marginality. If there exists a choice for the $\Theta^{I\bar{J}}$ components of the noncommutativity matrix and the parameter ρ which respects these requirements, we will be able to describe the deformation in noncommutative terms.⁴ We will see in the following that this is indeed the case here.

Let us first find out what are the possible (1, 1) pieces of Θ which respect the symmetries of the theory. Consider for instance the commutator $[z^1, \bar{z}^2] = i\Theta^{1\bar{2}}(z, \bar{z})$. We easily see that:

$$[z^1, \bar{z}^2] \xrightarrow{\mathbb{Z}_{3(2)}} e^{-\frac{i2\pi}{3}} [z^1, \bar{z}^2]. \quad (25)$$

Eq. (25) constrains $\Theta^{1\bar{2}}$ to either vanish or be a combination of any of the following: $\bar{z}^1 z^3, \bar{z}^3 z^2, z^1 \bar{z}^2$. All of the choices displayed are additionally invariant under the other discrete symmetry of the theory $\mathbb{Z}_{3(1)}$ as they should. Several possibilities exist for the rest of the components of $\Theta^{I\bar{J}}$ as well. In summary, the discrete global symmetries cannot completely fix the non-commutativity matrix. To determine Θ_ρ uniquely we transform Θ to spherical coordinates.⁵ Then we require it to be real, transverse to and independent of the radial direction r . The last requirement implements the exact marginality of the deformation in the dual description.

Imposing these constraints we find that there are just *two* distinct possibilities for $\Theta^{I\bar{J}}$. One of them is valid for $\rho \equiv -q_1 \in \mathbb{R}$:

$$\Theta_1 = iq_1 \begin{pmatrix} 0 & z_3^2 & -z_2^2 & 0 & -z_3 \bar{z}_1 + z_2 \bar{z}_3 & z_2 \bar{z}_1 - z_3 \bar{z}_2 \\ -z_3^2 & 0 & z_1^2 & \bar{z}_2 z_3 - z_1 \bar{z}_3 & 0 & -z_1 \bar{z}_2 + z_3 \bar{z}_1 \\ z_2^2 & -z_1^2 & 0 & -z_2 \bar{z}_3 + z_1 \bar{z}_2 & z_1 \bar{z}_3 - z_2 \bar{z}_1 & 0 \\ 0 & -z_3 \bar{z}_2 + z_1 \bar{z}_3 & z_2 \bar{z}_3 - z_1 \bar{z}_2 & 0 & -\bar{z}_3^2 & \bar{z}_2^2 \\ z_3 \bar{z}_1 - z_2 \bar{z}_3 & 0 & -z_1 \bar{z}_3 + z_2 \bar{z}_1 & \bar{z}_3^2 & 0 & -\bar{z}_1^2 \\ -z_2 \bar{z}_1 + z_3 \bar{z}_2 & z_1 \bar{z}_2 - z_3 \bar{z}_1 & 0 & -\bar{z}_2^2 & \bar{z}_1^2 & 0 \end{pmatrix} \quad (26)$$

⁴ Note for instance, that this description is not valid for the β -deformed theory when $\beta \in \mathbb{I}$.

⁵ Refer to Appendix A for the noncommutativity matrix in different coordinate systems.

and the other one, for $\rho \equiv iq_2$ with $q_2 \in \mathbb{R}$:

$$\Theta_2 = q_2 \begin{pmatrix} 0 & z_3^2 & -z_2^2 & 0 & z_3 \bar{z}_1 + z_2 \bar{z}_3 & -z_2 \bar{z}_1 - z_3 \bar{z}_2 \\ -z_3^2 & 0 & z_1^2 & -\bar{z}_2 z_3 - z_1 \bar{z}_3 & 0 & z_1 \bar{z}_2 + z_3 \bar{z}_1 \\ z_2^2 & -z_1^2 & 0 & z_2 \bar{z}_3 + z_1 \bar{z}_2 & -z_1 \bar{z}_3 - z_2 \bar{z}_1 & 0 \\ 0 & z_3 \bar{z}_2 + z_1 \bar{z}_3 & -z_2 \bar{z}_3 - z_1 \bar{z}_2 & 0 & \bar{z}_3^2 & -\bar{z}_2^2 \\ -z_3 \bar{z}_1 - z_2 \bar{z}_3 & 0 & z_1 \bar{z}_3 + z_2 \bar{z}_1 & -\bar{z}_3^2 & 0 & \bar{z}_1^2 \\ z_2 \bar{z}_1 + z_3 \bar{z}_2 & -z_1 \bar{z}_2 - z_3 \bar{z}_1 & 0 & \bar{z}_2^2 & -\bar{z}_1^2 & 0 \end{pmatrix} \quad (27)$$

Combining the two into $\Theta_\rho = \Theta_1 + \Theta_2$ we define a unique noncommutativity matrix Θ describing the ρ -deformation for general complex $\rho = (-q_1 + iq_2) \in \mathbb{C}$. This indicates that a noncommutative description of the transverse space is valid throughout the whole of the ρ parameter space, contrary to what happens for the β -deformed gauge theory.

Let us now discuss the properties of Θ_ρ . Recall that the noncommutativity parameter for the β -deformed theory, turned out to be position independent along isometry directions of the metric. This was crucial for employing the Lunin–Maldacena generating technique. We do not expect Θ_ρ to be constant along isometry directions since we know that the ρ -deformed theory does not respect any other global $U(1)$ symmetries except for the R-symmetry. Indeed, Θ_ρ is of a highly nontrivial form even when written in spherical coordinates (see [Appendix A](#)). However, it would be nice to find a coordinate system for which Θ_ρ is position independent, even if not along isometry directions.⁶

It is a curious fact that Θ_β defined in (12) satisfies the following two conditions:

$$\begin{aligned} \text{Divergence free condition: } & \partial_i \Theta^{ij} = 0 \\ \text{Associativity condition: } & T^{[ijk]} \equiv \Theta^{il} \partial_l \Theta^{jk} + \Theta^{kl} \partial_l \Theta^{ij} + \Theta^{jl} \partial_l \Theta^{ki} = 0 \end{aligned} \quad (28)$$

which also imply that $T^{[ijk]} = \partial_l (\Theta^{li} \Theta^{jk}) = 0$.

It is easy to see that Θ_ρ satisfies the first condition but fails to preserve the associativity constraint. This is disconcerting because it is not clear whether nonassociative deformations can be described through modified star products. As a result it is far from obvious whether we can rewrite the Lagrangian of the ρ -deformed gauge theory as that of the $\mathcal{N} = 4$ Lagrangian with the usual product between the matter content of the theory replaced by some star product. Furthermore, a coordinate system in which Θ_ρ is constant does not exist (contrary to what happens for the β -deformation).⁷

Finally, it is worth mentioning that the failure of associativity has its roots in the (1, 1) parts of the noncommutativity matrix, thus challenging our method for determining them. There exists however a rather non-trivial check that we have constructed the correct Θ describing the deformation. We saw in the previous section, that for some special points in the space of couplings of the marginally deformed theory, one can move from a theory where either β or ρ (but not both) is turned on, to a theory where both couplings are nonvanishing. The whole analysis as well as the appropriate field redefinitions which took us from one point to the other in the deformation space, relied on the holomorphicity of the superpotential. It would thus appear quite improbable that we would be able to see it happening in this context. In principle however, one would expect that if the deformation is indeed described from an open string theory perspective as a noncommutative deformation of the transverse space, then at these special points Θ should transform under a change of coordinates from Θ_β or Θ_ρ to $\Theta = \Theta_{\tilde{\beta}} + \Theta_{\tilde{\rho}}$. Moreover, one might

⁶ This is the case for the nongeometric Q-space [99,30,68], for instance.

⁷ This does not exclude the possibility of finding a reference frame for which Θ_ρ is position independent. Integrability however will be lost.

hope that the coordinate transformation which would make this possible would be the precise analog of the field redefinition applied to the gauge theory. Note however that in the case of the β -deformation, it is only for $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ that a noncommutative description — with parameter $a = 2 \sin \beta_{\mathbb{R}}$ — is valid. This implies that we can apply the above consistency check if and only if both the original and final points in the coupling constant space involve a real parameter $\beta_{\mathbb{R}}$. A glance at the previous section will convince us that this indeed occurs: starting with $\rho = q_1 \in \mathbb{R}$ and $\beta = 0$ one can reach a point with $\tilde{\rho} = \frac{iq_1}{\sqrt{3}} \in \mathbb{I}$ and $\tilde{a} = 2 \sin \tilde{\beta} = \frac{2q_1}{\sqrt{3}} \in \mathbb{R}$.⁸ In fact, it is quite straightforward to check that a coordinate transformation according to (19) leads us from $\Theta_\rho = -\Theta_1$ to $\Theta = \Theta_{\tilde{a}=\frac{2q_1}{\sqrt{3}}} + \Theta_{\tilde{\rho}=\frac{iq_1}{\sqrt{3}}}$. Furthermore, it appears that this case exhausts all possible coordinate changes that relate noncommutativity matrices corresponding to different parameters of the Leigh–Strassler deformation. We take this result as evidence that both our prescription for determining the $(1, 1)$ parts of Θ as well as the very interpretation of the deformation in noncommutative terms are indeed justified.

4. The Seiberg–Witten equations and the deformed flat space solution

In the previous section, we saw how the deformation of the superpotential affects the moduli space of the gauge theory at large N . In particular, the six dimensional flat space with metric \mathcal{G}_{IJ} of the $\mathcal{N} = 4$ theory is promoted to a noncommutative space characterized now by the set \mathcal{G}_{IJ} and Θ^{IJ} . Both metric and noncommutativity parameter are mainly determined from the Lagrangian of the theory; the former is read off from the kinetic term of the scalars while the latter from their potential.

Since an $SU(N)$ gauge theory can be realized as the low energy limit of open strings attached on a stack of D3-branes, the set $(\mathcal{G}_{flat}, \Theta)$ describes the geometry of the transverse space as seen by the open strings in the limit of large N and $\alpha' \rightarrow 0$. We will thus refer to $(\mathcal{G}_{flat}, \Theta)$ as the *open* string parameters.

On the other hand, any theory of open strings necessarily contains closed strings. Closed strings however perceive the geometry quite differently from open strings. In fact, it was shown in [98,96] that target space noncommutativity from the point of view of open strings corresponds to turning on a B-field from the viewpoint of closed strings. The set (g, B) , with g the closed string metric, are the *closed* string parameters that describe the same geometry. In this context, (g, B) represent the deformed flat space solution into which D3-branes are immersed.⁹

Suppose now that we are given a set of equations relating the two groups of data. Then — provided that the open string parameters determined in the previous section exactly and fully describe the deformation — we could specify the closed string fields (g, B) of the deformed flat space geometry for free, i.e. without having to solve the type IIB differential equations of motion [97].

Equations relating open and closed string parameters indeed exist in the literature [18,1,96,98]:

$$g + B = \frac{1}{\mathcal{G}^{-1} + \Theta}$$

⁸ We are here using the result of Eq. (18) approximated for $\beta \ll 1$. The reason is that the non-commutativity matrix for the β -deformation is valid only for small β as shown in [62] and at the end of Section 2.1.

⁹ We are obviously interested here in the limit where open and closed strings are decoupled from each other.

$$g_s = G_s \sqrt{\frac{\det \mathcal{G}^{-1}}{\det (\mathcal{G}^{-1} + \Theta)}} = G_s \sqrt{\frac{1}{\det (1 + \Theta \mathcal{G})}} \quad (29)$$

where G_s , g_s denote the corresponding open/closed string couplings.¹⁰ They were however considered in a situation somewhat different from the one discussed in this article, namely for a flat D-brane embedded in flat background space with a constant B-field turned on along its worldvolume [96,98]. It was under these circumstances that, the presence of the background B-field was shown to deform the algebra of functions on the worldvolume of the brane into that of a noncommutative Moyal type of algebra, where Θ is a c-number. While it is natural to ask what happens in situations where the B-field is not constant, technical difficulties have hindered progress in this direction. In the order of increasing complexity, two cases can be considered: the case of a closed $dB = 0$ though not necessarily constant two-form field B and the case of nonvanishing NS–NS three form flux $H = dB$ in a curved background. In [20] the former case was explored and the Moyal deformation of the algebra of functions on the brane worldvolume was shown to naturally extended to the Kontsevich star product deformation [61]. The authors of [23] — see also [51, 52,54] — undertook the study of the most general case where $H = dB \neq 0$. They considered a special class of closed string backgrounds, called parallelizable, and expanded the background fields in Taylor series. It was then possible to perturbatively analyze n -point string amplitudes on the disk and obtain — in a first order expansion — the appropriate generalization of (29). In fact, it turned out that Eq. (29) is still valid for a weakly varying nonclosed B-field even though the corresponding algebra of functions is now both noncommutative and nonassociative.

In this article, we want to apply the above formulas in a situation where the B-field lies in the *transverse* space to the D3-brane. This case has not been explicitly studied in the literature¹¹ but one expects by T-duality that Eqs. (29) should continue to hold. The most obvious concern here is that we do not have a set of conditions on the validity of (29) from the open string data. We have a non-commutative parameter which does not respect associativity and we have no way of knowing whether the corresponding B-field would be slowly varying or not. Nevertheless, if the general reasoning is correct and (29) indeed provide the relation between open and closed string parameters in this setup, the resulting closed string fields (g_s, g, B) will constitute a new supergravity solution, i.e. the deformed flat space solution where D3-branes should be embedded.

A natural place to test these thoughts first is the β -deformed theory for which both the gravity dual and the corresponding deformed flat space solution are known [69]. The open string data $(\mathcal{G}_{flat}, \Theta_{\beta_{\mathbb{R}}})$ describing the β -deformation are given in Section 2. In this case the noncommutativity parameter Θ_{β} turned out to be position independent although the associated NS–NS three form flux was non-zero. It is easy to show that applying (29) to the open string parameters $(\mathcal{G}_{flat}, \Theta_{\beta_{\mathbb{R}}})$ one recovers the deformed flat space geometry found by Lunin and Maldacena in [69]. This follows trivially from the fact that Eq. (29) and the T-duality transformation rules of (8) are identical; yet the interpretation of the variables involved is different. We will return to this point again in the following section.

To proceed, we check whether the closed string fields (g_s, g, B) determined from (29) for the ρ -deformation, satisfy the supergravity equations of motion. It turns out that they do but only up to *third* order in the deformation parameter ρ . The discrepancy at higher orders is expected since there is no way to determine the validity of (29) for nonassociative deformations. At the same time, nonassociativity becomes manifest at second order in the deformation parameter. We

¹⁰ Note that $G_s = 1$ for the ρ -deformation.

¹¹ Mainly because a constant B-field in the transverse space can be gauged away leaving no trace on the geometry.

postpone further discussion on this issue until Section 6 and close this section with the explicit solution to this order.

The dilaton is given by

$$e^{2\Phi} = G$$

$$G = 1 + r_1^2[(q_1 y - q_2 x_1)^2 + (q_1 y_1 - q_2 x)^2 + (q_1 x_3 - q_2 y_2)^2 + (q_1 x_2 - q_2 y_3)^2] \\ + r_2^2[(q_1 y - q_2 x_2)^2 + (q_1 y_2 - q_2 x)^2 + (q_1 x_1 - q_2 y_3)^2 + (q_1 x_3 - q_2 y_1)^2] \\ + r_3^2[(q_1 y - q_2 x_3)^2 + (q_1 x_2 - q_2 y_1)^2 + (q_1 x_1 - q_2 y_2)^2 + (q_1 y_3 - q_2 x)^2] \quad (30)$$

Here and in the following expressions, $\rho \equiv -2q_1 + i2q_2$ and x, x_i, y, y_i are defined as

$$x_1 = -C_1 r_1 + C_2 r_2 + C_3 r_3, \quad x_2 = C_1 r_1 - C_2 r_2 + C_3 r_3, \\ x_3 = C_1 r_1 + C_2 r_2 - C_3 r_3 \\ y_1 = -S_1 r_1 + S_2 r_2 + S_3 r_3, \quad y_2 = S_1 r_1 - S_2 r_2 + S_3 r_3, \quad y_3 = S_1 r_1 + S_2 r_2 - S_3 r_3 \\ x = C_1 r_1 + C_2 r_2 + C_3 r_3, \quad y = S_1 r_1 + S_2 r_2 + S_3 r_3 \quad (31)$$

where S_i, C_i represent the following trigonometric functions:

$$S_1 = \sin(\varphi_2 + \varphi_3 - 2\varphi_1), \quad S_2 = \sin(\varphi_3 + \varphi_1 - 2\varphi_2), \quad S_3 = \sin(\varphi_1 + \varphi_2 - 2\varphi_3) \\ C_1 = \cos(\varphi_2 + \varphi_3 - 2\varphi_1), \quad C_2 = \cos(\varphi_3 + \varphi_1 - 2\varphi_2), \quad C_3 = \cos(\varphi_1 + \varphi_2 - 2\varphi_3) \quad (32)$$

Using the definitions above, we can write the B-field as

$$B_{r_1 r_2} = r_3(q_2 x_3 - q_1 y), \quad B_{r_2 r_3} = r_1(q_2 x_1 - q_1 y), \quad B_{r_3 r_1} = r_2(q_2 x_2 - q_1 y) \\ B_{r_1 \varphi_2} = -r_2 r_3(q_1 x_2 - q_2 y_1), \quad B_{r_1 \varphi_3} = r_2 r_3(q_1 x_3 - q_2 y_1) \\ B_{r_2 \varphi_1} = r_1 r_3(q_1 x_1 - q_2 y_2), \quad B_{r_2 \varphi_3} = -r_1 r_3(q_1 x_3 - q_2 y_2) \\ B_{r_3 \varphi_1} = -r_1 r_2(q_1 x_1 - q_2 y_3), \quad B_{r_3 \varphi_2} = r_1 r_2(q_1 x_2 - q_2 y_3) \\ B_{\varphi_1 \varphi_2} = r_1 r_2 r_3(q_2 x - q_1 y_3), \quad B_{\varphi_2 \varphi_3} = r_1 r_2 r_3(q_2 x - q_1 y_1) \\ B_{\varphi_3 \varphi_1} = r_1 r_2 r_3(q_2 x - q_1 y_2). \quad (33)$$

Finally, for the metric components we find the following complicated expressions:

$$g_{r_1 r_1} = 1 - (r_2^2[(q_1 x_3 - q_2 y_1)^2 + (q_1 y - q_2 x_2)^2] \\ + r_3^2[(q_1 y - q_2 x_3)^2 + (q_1 x_2 - q_2 y_1)^2]) \\ g_{r_2 r_2} = 1 - (r_1^2[(q_1 x_3 - q_2 y_2)^2 + (q_1 y - q_2 x_1)^2] \\ + r_3^2[(q_1 y - q_2 x_3)^2 + (q_1 x_1 - q_2 y_2)^2]) \\ g_{r_3 r_3} = 1 - (r_1^2[(q_1 x_2 - q_2 y_3)^2 + (q_1 y - q_2 x_1)^2] \\ + r_2^2[(q_1 y - q_2 x_2)^2 + (q_1 x_1 - q_2 y_3)^2]) \\ g_{\varphi_1 \varphi_1} = r_1^2[1 - (r_2^2[(q_1 x_1 - q_2 y_3)^2 + (q_1 y_2 - q_2 x)^2] \\ + r_3^2[(q_1 y_3 - q_2 x)^2 + (q_1 x_1 - q_2 y_2)^2])] \\ g_{\varphi_2 \varphi_2} = r_2^2[1 - (r_1^2[(q_1 x_2 - q_2 y_3)^2 + (q_1 y_1 - q_2 x)^2] \\ + r_3^2[(q_1 y_3 - q_2 x)^2 + (q_1 x_2 - q_2 y_1)^2])] \\ g_{\varphi_3 \varphi_3} = r_3^2[1 - (r_1^2[(q_1 x_1 - q_2 y_2)^2 + (q_1 y_3 - q_2 x)^2] \\ + r_2^2[(q_1 y_2 - q_2 x)^2 + (q_1 x_3 - q_2 y_1)^2])]$$

$$\begin{aligned}
g_{\varphi_3\varphi_3} &= r_3^2 [1 - (r_1^2 [(q_1x_3 - q_2y_2)^2 + (q_1y_1 - q_2x)^2] \\
&\quad + r_2^2 [(q_1y_2 - q_2x)^2 + (q_1x_3 - q_2y_1)^2])] \\
g_{r_1r_2} &= r_1r_2 [(q_1x_3 - q_2y_1)(q_1x_3 - q_2y_2) + (q_1y - q_2x_1)(q_1y - q_2x_2)] \\
g_{r_1r_3} &= r_1r_3 [(q_1x_2 - q_2y_1)(q_1x_2 - q_2y_3) + (q_1y - q_2x_3)(q_1y - q_2x_1)] \\
g_{r_1\varphi_1} &= r_1 (r_2^2 [(q_1x_3 - q_2y_1)(q_2x - q_1y_2) + (q_2x_2 - q_1y)(q_1x_1 - q_2y_3)] \\
&\quad + r_3^2 [(q_2x_3 - q_1y)(q_1x_1 - q_2y_2) + (q_1x_2 - q_2y_1)(q_2x - q_1y_3)]) \\
g_{r_1\varphi_2} &= r_1r_2^2 [(q_2x - q_1y_1)(-q_1x_3 + q_2y_1) + (q_2x_2 - q_1y)(-q_1x_2 + q_2y_3)] \\
g_{r_1\varphi_3} &= r_1r_3^2 [(q_2x - q_1y_1)(-q_1x_2 + q_2y_1) + (q_2x_3 - q_1y)(-q_1x_3 + q_2y_2)] \\
g_{r_2r_3} &= r_2r_3 [(q_1x_1 - q_2y_3)(q_1x_1 - q_2y_2) + (q_1y - q_2x_3)(q_1y - q_2x_2)] \\
g_{r_2\varphi_1} &= r_2r_1^2 [(q_2x - q_1y_2)(-q_1x_3 + q_2y_2) + (q_2x_1 - q_1y)(-q_1x_1 + q_2y_3)] \\
g_{r_2\varphi_2} &= r_2 (r_1^2 [(q_1x_3 - q_2y_2)(q_2x - q_1y_1) + (q_2x_1 - q_1y)(q_1x_2 - q_2y_3)] \\
&\quad + r_3^2 [(q_2x_3 - q_1y)(q_1x_2 - q_2y_1) + (q_1x_1 - q_2y_2)(q_2x - q_1y_3)]) \\
g_{r_2\varphi_3} &= r_2r_3^2 [(q_2x_3 - q_1y)(-q_1x_3 + q_2y_1) + (q_2x - q_1y_2)(-q_1x_1 + q_2y_2)] \\
g_{r_3\varphi_1} &= r_3r_1^2 [(q_2x_1 - q_1y)(-q_1x_1 + q_2y_2) + (q_2x - q_1y_3)(-q_1x_2 + q_2y_3)] \\
g_{r_3\varphi_2} &= r_3r_2^2 [(q_2x_2 - q_1y)(-q_1x_2 + q_2y_1) + (q_2x - q_1y_3)(-q_1x_1 + q_2y_3)] \\
g_{r_3\varphi_3} &= r_3 (r_2^2 [(q_1x_3 - q_2y_1)(q_2x_2 - q_1y) + (q_2x - q_1y_2)(q_1x_1 - q_2y_3)] \\
&\quad + r_1^2 [(q_2x_1 - q_1y)(q_1x_3 - q_2y_2) + (q_1x_2 - q_2y_3)(q_2x - q_1y_1)]) \\
g_{\varphi_1\varphi_2} &= r_1^2r_2^2 [(q_2x - q_1y_1)(-q_1x + q_2x) + (q_2y_3 - q_1x_2)(-q_1x_1 + q_2y_3)] \\
g_{\varphi_1\varphi_3} &= r_1^2r_3^2 [(q_2x - q_1y_1)(-q_1y_3 + q_2x) + (q_2y_2 - q_1x_3)(-q_1x_1 + q_2y_2)] \\
g_{\varphi_2\varphi_3} &= r_2^2r_3^2 [(q_2x - q_1y_2)(-q_1y_3 + q_2x) + (q_2y_1 - q_1x_3)(-q_1x_2 + q_2y_1)]. \tag{34}
\end{aligned}$$

5. D-branes in deformed $\text{AdS}_5 \times \text{S}^5$ and the near horizon geometry

In this section we proceed to determine the gravity dual of the ρ -deformed gauge theory up to third order in the deformation parameter. Let us first discuss what happens in the case of the β -deformation where the dual geometry is known. In the previous section we observed that the T-duality transformation rules (8) with which the Lunin–Maldacena solution was constructed, are identical in form to (29). To obtain the dual geometry for the β -deformation we saw in Section 2 that we must use:

$$E_0 = g_{\text{AdS}_5 \times \text{S}^5} \quad \text{and} \quad \Gamma = \Theta_{\beta_{\mathbb{R}}} \tag{35}$$

Suppose now that we want to interpret these variables according to (29). We would think of $g_{\text{AdS}_5 \times \text{S}^5}$ as the open string metric $\mathcal{G}_{\text{AdS}_5 \times \text{S}^5}$ whereas of Γ as $\Theta_{\beta_{\mathbb{R}}}$. In this sense, $(G_s = g_{\text{YM}}^2, \mathcal{G}_{\text{AdS}_5 \times \text{S}^5}, \Theta_{\beta_{\mathbb{R}}})$ would encode the geometry as seen at large N by the open strings attached on a D3-brane embedded in the Lunin–Maldacena (11) background. In the following we denote the NS–NS fields of the solution as $(\tilde{g}_s, \tilde{g}, \tilde{B})$.

In other words, consider a stack of N D3-branes in the deformed flat space geometry of (10). The near horizon limit of this configuration is the gravity dual of the Leigh–Strassler marginal deformation with $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$ and $\rho = 0$. A probe D3-brane propagating near the stack will

then be described by the DBI action written either in terms of the closed $(\tilde{g}_s, \tilde{g}, \tilde{B})$ or of the open $(G_s, \mathcal{G}_{\text{AdS}_5 \times S^5}, \Theta_\beta)$ string fields. However, the action of a single D3-brane separated from a collection of $(N - 1)$ other branes can also be obtained by integrating out the massive open strings stretched between the probe and the source. Indeed, as expected according to [73,71,106,80,22,8], the DBI action describing the motion of a D3-brane in this background should in the large N limit coincide with the leading IR part of the quantum effective action of the β -deformed theory obtained by keeping the $U(1)$ external fields and integrating over the massive ones.

In this spirit, it does not seem surprising that the appropriate open string data are the metric of $\text{AdS}_5 \times S^5$ and the noncommutativity parameter $\Theta_{\beta_{\mathbb{R}}}$. In fact, the action of the β -deformed gauge theory can be written as that of the parent $\mathcal{N} = 4$ theory with the product of the matter fields replaced by a star product associated to $\Theta_{\beta_{\mathbb{R}}}$. Moreover, as conjectured in [69,78,65] and later proven in [58], all planar amplitudes are equal to their $\mathcal{N} = 4$ counterparts up to an overall phase factor. This suggests that the iterative structure of the large N β -deformed gauge theory amplitudes, when $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$, is identical to that of the $\mathcal{N} = 4$ SYM theory. It is then not hard to imagine that the quantum effective action mentioned above will be analogous to that of the undeformed theory with the only difference being some phase factors coming from the noncommutative deformation of the product. Subsequently, the open string fields appearing in the DBI form of the effective action of the $\mathcal{N} = 4$ theory $(G_s, \mathcal{G}, \Theta = 0)$ will be promoted to $(G_s, \mathcal{G}, \Theta_{\beta_{\mathbb{R}}})$.

It is natural to wonder whether a similar situation could apply to the ρ -deformation as well. The results of Section 3 suggest that this is likely not the case. Suppose we succeeded in writing the action of the ρ -deformed theory as the $\mathcal{N} = 4$ action with a star product between the matter fields. It would still be difficult to understand how planar equivalence between the two theories would be achieved given that the deformation is both noncommutative and nonassociative. In fact, the proof given in [58] specifically relied on the associativity of the star product for the β -deformation. However, nonassociativity shows up at second order in ρ and in view of the results of the previous section one may hope that a solution to this order could be obtained here too.

To explicitly check if this is the case, we can use the second order expansion of (29):

$$\begin{aligned} g &= \mathcal{G} + \mathcal{G}\Theta\mathcal{G}\Theta\mathcal{G} + \mathcal{O}(\rho^4) \\ B &= -\mathcal{G}\Theta\mathcal{G} + \mathcal{O}(\rho^3) \\ G^{-1} &= 1 + \text{Tr}\left[\mathcal{G}\Theta - \frac{1}{2}\mathcal{G}\Theta\mathcal{G}\Theta\right] + \mathcal{O}(\rho^4), \end{aligned} \quad (36)$$

where \mathcal{G} is here the metric of $\text{AdS}_5 \times S^5$ and $\Theta = \Theta_\rho$ defined in Section 3. Eqs. (36) relate the open string parameters of the deformed theory with the NS–NS string fields of the dual geometry. To find the RR-fluxes we resort to the type IIB equations of motion. We refer the reader to Appendix B for the necessary definitions of the parameters involved as well as the type IIB field equations [97] in five dimensions.

We assume that there is no warp factor in front the metric to this order and make the standard ansatz for the five form field strength

$$\begin{aligned} ds_{10}^2 &= ds_{\text{AdS}_5}^2 + ds_{S^5}^2 \\ F_5 &= f(\omega_{\text{AdS}_5} + \omega_{S^5}). \end{aligned} \quad (37)$$

Here f is the normalization coefficient for the flux equal to $\tilde{f} = 16\pi N$, and ω_{AdS_5} , $\omega_{\tilde{S}^5}$ are the volume elements of the corresponding parts of the $\text{AdS}_5 \times \tilde{S}^5$ geometry. Eq. (37) allows us to solve for the RR three form flux F_3 :

$$\begin{aligned} F_3 &= -f^{-1} d \star_5 e^{-2\Phi} H_3 \\ H_3 &= f^{-1} d \star_5 F_3 \Rightarrow d[B - f^{-1} \star_5 F_3] = 0. \end{aligned} \quad (38)$$

Note that to this order $F_3 = f \star_5 B$ which greatly simplifies calculations. This relation is mainly due to $\delta_{S^5} \Omega_\rho = 0$ where $\Omega_\rho = \mathcal{G} \Theta \mathcal{G}$ denotes the form on S^5 associated to the bivector Θ_ρ . It is clear from (36) that $B = -\Omega$ to this order. It is worth remarking that $F_3 = f \star B$ is exact for the β -deformed theory where both $d\Omega_\beta = 0$ and $\delta_{S^5} \Omega_\beta = 0$ hold. It is then possible to show that the type IIB equations are simultaneously satisfied up to third order in the deformation parameter, for the following set of fields¹².

The F_3 and F_5 -form flux are simply given by

$$\begin{aligned} F_3 &= \star_{S^5} \Omega \quad \text{with } \Omega \equiv \mathcal{G}_{ik} \mathcal{G}_{jl} \Theta^{kl} dx^i \wedge dx^j \\ F_5 &= f(\omega_{\text{AdS}_5} + G\omega_{S^5}), \end{aligned} \quad (39)$$

with \mathcal{G} the metric of $\text{AdS}_5 \times S_5$.

The dilaton, on the other hand, can be expressed as follows:

$$\begin{aligned} e^{2\Phi} &= e^{2\Phi_0} G \\ G^{-1} &= 1 + q_1^2 (v^2 + s_\alpha^2 u_1^2 + (c_\alpha^2 + s_\alpha^2 c_\theta^2) u_2^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2) u_3^2 + c_\alpha^2 v_1^2 + s_\alpha^2 s_\theta^2 v_2^2 + s_\alpha^2 c_\theta^2 v_3^2) \\ &\quad + 2q_1 q_2 (uv + (c_\alpha^2 - s_\alpha^2) u_1 v_1 - (c_\theta^2 + s_\theta^2 (c_\alpha^2 - s_\alpha^2)) u_2 v_2 - (s_\theta^2 + c_\theta^2 (c_\alpha^2 - s_\alpha^2)) \\ &\quad + q_2^2 (u^2 + c_\alpha^2 u_1^2 + s_\alpha^2 s_\theta^2 u_2^2 + s_\alpha^2 c_\theta^2 u_3^2 + s_\alpha^2 v_1^2 + (c_\alpha^2 + s_\alpha^2 c_\theta^2) v_2^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2) v_3^2), \end{aligned} \quad (40)$$

where to keep the expression compact we defined

$$\begin{aligned} u_1 &= (-c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3), & v_1 &= (-c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \\ u_2 &= (c_\alpha C_1 - s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3), & v_2 &= (c_\alpha S_1 - s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \\ u_3 &= (c_\alpha C_1 + s_\alpha s_\theta C_2 - s_\alpha c_\theta C_3), & v_3 &= (c_\alpha S_1 + s_\alpha s_\theta S_2 - s_\alpha c_\theta S_3) \\ v &= (c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3), & u &= (c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3) \end{aligned} \quad (41)$$

with (S_i, C_i) the trigonometric functions defined in (32) and $(c_\alpha, s_\alpha, c_\theta, s_\theta) \equiv (\cos \alpha, \sin \alpha, \cos \theta, \sin \theta)$ so that the parametrization of the deformed five-sphere is given in terms of the angular variables $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$.

Using the same notations we write the components of the B-field as

$$\begin{aligned} B_{\alpha\theta} &= s_\alpha (q_1 v + q_2 u), & B_{\alpha\varphi_1} &= 0 \\ B_{\alpha\varphi_2} &= s_\alpha s_\theta c_\theta (q_1 u_2 - q_2 v_2), & B_{\alpha\varphi_3} &= s_\alpha s_\theta c_\theta (-q_1 u_3 + q_2 v_3) \\ B_{\theta\varphi_1} &= c_\alpha s_\alpha^2 (q_1 u_1 - q_2 v_1), & B_{\theta\varphi_2} &= -c_\alpha s_\alpha^2 s_\theta^2 (q_1 u_2 - q_2 v_2) \\ B_{\theta\varphi_3} &= -c_\alpha s_\alpha^2 c_\theta^2 (q_1 u_3 - q_2 v_3), & B_{\varphi_1\varphi_2} &= -c_\alpha s_\alpha^2 s_\theta c_\theta (q_1 v_3 + q_2 u_3) \\ B_{\varphi_2\varphi_3} &= -c_\alpha s_\alpha^2 s_\theta c_\theta (q_1 v_1 + q_2 u_1), & B_{\varphi_3\varphi_1} &= -c_\alpha s_\alpha^2 s_\theta c_\theta (q_1 v_2 + q_2 u_2). \end{aligned} \quad (42)$$

¹² We set $R = 1$ where R the radius of AdS_5 .

Finally, we give the components of the metric g on the deformed sphere:

$$\begin{aligned}
g_{\alpha\alpha} &= 1 - q_1^2(c_\theta^2 u_2^2 + s_\theta^2 u_3^2 + v^2) + 2q_1 q_2(-uv + c_\theta^2 u_2 v_2 + s_\theta^2 u_3 v_3) \\
&\quad - q_2^2(u^2 + c_\theta^2 v_2^2 + s_\theta^2 v_3^2) \\
g_{\theta\theta} &= s_\alpha^2[1 - q_1^2(v^2 + s_\alpha^2 u_1^2 + c_\alpha^2(s_\theta^2 u_2^2 + c_\theta^2 u_3^2)) \\
&\quad - 2q_1 q_2(uv - s_\alpha^2 u_1 v_1 - c_\alpha^2(s_\theta^2 u_2 v_2 + c_\theta^2 u_3 v_3)) \\
&\quad - q_2^2(u^2 + s_\alpha^2 v_1^2 + c_\alpha^2(s_\theta^2 v_2^2 + c_\theta^2 v_3^2))] \\
g_{\varphi_1\varphi_1} &= c_\alpha^2[1 - s_\alpha^2(q_1^2(u_1^2 + s_\theta^2 v_2^2 + c_\theta^2 v_3^2) + 2q_1 q_2(-u_1 v_1 + s_\theta^2 u_2 v_2 + c_\theta^2 u_3 v_3) \\
&\quad + q_2^2(v_1^2 + s_\theta^2 u_2^2 + c_\theta^2 u_3^2))] \\
g_{\varphi_2\varphi_2} &= s_\alpha^2 s_\theta^2[1 - q_1^2(u_2^2(c_\alpha^2 + s_\alpha^2 c_\theta^2) + c_\alpha^2 v_1^2 + s_\alpha^2 c_\theta^2 v_3) \\
&\quad + 2q_1 q_2(-c_\alpha^2 u_1 v_1 + (c_\alpha^2 + s_\alpha^2 c_\theta^2)u_2 v_2 - s_\alpha^2 c_\theta^2 u_3^2 v_3^2) \\
&\quad - q_2^2(c_\alpha^2 u_1^2 + s_\alpha^2 c_\theta^2 u_3^2 + (c_\alpha^2 + s_\alpha^2 c_\theta^2)v_2^2)] \\
g_{\varphi_3\varphi_3} &= s_\alpha^2 c_\theta^2[1 - q_1^2(c_\alpha^2 v_1^2 + s_\alpha^2 s_\theta^2 v_2^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2)u_2^2) \\
&\quad + 2q_1 q_2(-c_\alpha^2 u_1 v_1 - s_\alpha^2 s_\theta^2 u_2 v_2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2)u_3 v_3) \\
&\quad - q_2^2(c_\alpha^2 u_1^2 + s_\alpha^2 s_\theta^2 u_2^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2)v_3^2)] \\
g_{\alpha\theta} &= c_\alpha s_\alpha c_\theta s_\theta[q_1^2(u_2^2 - u_3^2) + 2q_1 q_2(-u_2 v_2 + u_3 v_3) + q_2^2(v_2^2 - v_3^2)] \\
g_{\alpha\varphi_1} &= c_\alpha s_\alpha[q_1^2(u_1 v + s_\theta^2 u_3 v_2 c_\theta^2 u_2 v_3) + q_1 q_2(uu_1 - vv_1 + u_2 u_3 - v_2 v_3) \\
&\quad - q_2^2(uv_1 + c_\theta^2 u_3 v_2 + s_\theta^2 u_2 v_3)] \\
g_{\alpha\varphi_2} &= c_\alpha s_\alpha s_\theta^2[-q_1^2(u_2 v + u_3 v_1) + q_1 q_2(-uu_2 - u_1 u_3 + vv_2 + v_1 v_3) + q_2^2(uv_2 + u_1 v_3)] \\
g_{\alpha\varphi_3} &= c_\alpha s_\alpha s_\theta^2[-q_1^2(u_2 v + u_3 v_1) + q_1 q_2(-uu_2 - u_1 u_3 + vv_2 + v_1 v_3) + q_2^2(uv_2 + u_1 v_3)] \\
g_{\theta\varphi_1} &= c_\alpha^2 s_\alpha^2 c_\theta s_\theta(q_1^2 + q_2^2)(u_3 v_2 - u_2 v_3) \\
g_{\theta\varphi_2} &= s_\alpha^2 s_\theta c_\theta[q_2^2(uv_2 + s_\alpha^2 u_3 v_1 + c_\alpha^2 u_1 v_3) - q_1 q_2(uu_2 - vv_2 + u_1 u_3 - v_1 v_3) \\
&\quad - q_1^2(u_1 v + c_\alpha^2 u_3 v_1 + s_\alpha^2 u_1 v_3)] \\
g_{\theta\varphi_3} &= s_\alpha^2 s_\theta c_\theta[q_1^2(u_3 v + c_\alpha^2 u_2 v_1 + s_\alpha^2 u_1 v_2) + q_1 q_2(u_1 u_2 - v_1 v_2 + uu_3 - vv_3) \\
&\quad - q_2^2(uv_3 + s_\alpha^2 u_2 v_1 + c_\alpha^2 u_1 v_2)] \\
g_{\varphi_1\varphi_2} &= c_\alpha^2 s_\alpha^2 s_\theta^2(q_1^2 + q_2^2)(u_1 u_2 + v_1 v_2) \\
g_{\varphi_2\varphi_3} &= c_\alpha^2 s_\alpha^2 c_\theta^2(q_1^2 + q_2^2)(u_1 u_3 + v_1 v_3) \\
g_{\varphi_3\varphi_1} &= s_\alpha^4 c_\theta^2 s_\theta^2(q_1^2 + q_2^2)(u_2 u_3 + v_2 v_3).
\end{aligned} \tag{43}$$

6. Discussion

In this article we studied the Leigh–Strassler marginal deformation of $\mathcal{N} = 4$ SYM for $\rho \neq 0$. We made precise the relation of the deformation to noncommutativity by constructing a non-commutativity matrix Θ_ρ which describes it. We then considered Θ_ρ as part of the open string data pertaining to the theory and used the Seiberg–Witten relations to obtain the corresponding closed string data (Fig. 1). We were thus able to find supergravity solutions corresponding to the

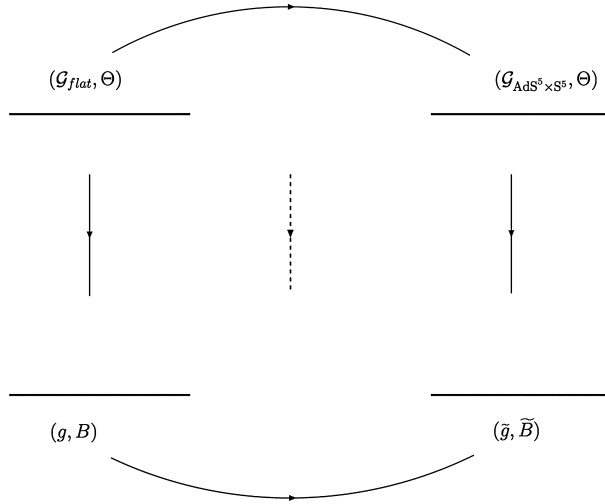


Fig. 1. The mapping between open and closed string parameters.

flat space deformation and the AdS/CFT dual of the deformed theory, up to third order in the deformation parameter.

The noncommutativity matrix Θ_ρ is crucially different from Θ_β , the noncommutativity matrix describing the β -deformation, in its failure to preserve the property of associativity. The lack of associativity makes the possibility of defining a star product dubious. As a result, the Lagrangian of the ρ -deformed theory cannot be readily expressed in terms of the $\mathcal{N} = 4$ SYM Lagrangian with a modified product between the matter fields.

Similar issues arise when one considers the D-terms of the potential. It is possible to rewrite the D-terms of the $\mathcal{N} = 4$ theory as a sum of the F-terms plus an additional potential term involving the commutator between holomorphic and antiholomorphic matter fields:

$$\text{Tr}[\Phi_I, \tilde{\Phi}^I][\Phi_J, \tilde{\Phi}^J] = \text{Tr}[\Phi_I, \Phi_J][\tilde{\Phi}^I, \tilde{\Phi}^J] + \text{Tr}[\Phi_I, \tilde{\Phi}^J][\Phi_J, \tilde{\Phi}^I] \quad (44)$$

It is clear from (44) that should we wish to deform only the F-terms of the potential, we must appropriately alter the commutator: $[\Phi_I, \tilde{\Phi}^J]$. For the β -deformed gauge theory, the $(1, 1)$ pieces of the noncommutativity matrix ensured that the D-terms remained unaffected by the deformation according to (44). The lack of a star product in the case of the ρ -deformation however, makes it impossible to perform this consistency check.

Regarding the mapping between open and closed string fields; it is clear that Eqs. (29) in Sections 4 and 5 are not valid in this case, especially due to the nonassociativity of Θ_ρ . It seems natural to expect that when T^{ijk} of (28) is nonvanishing, both Θ and $T = \Theta \partial \Theta$ are necessary for defining the deformation. A natural generalization of (29) would then relate $(\mathcal{G}, \Theta, T = \Theta \partial \Theta)$ to $(g, B, H = dB)$ and perhaps provide the deformed flat space solution to all orders in the deformation.

Even if finding the appropriate mapping between open and closed string fields might help obtain the deformed flat space geometry, it would not necessarily solve the problem of finding the dual gravity background as well. It is possible that nonassociativity spoils the planar equivalence between the $\mathcal{N} = 4$ theory and its deformation. This would obviously be reflected on the form of the quantum effective action and therefore of the DBI, making it difficult to determine the relevant open string data.

So far we have considered the Leigh–Strassler marginal deformation at the point $\beta = 0$. However, quantum corrections will probably generate a β -like term since a symmetry argument does not prohibit it. In this sense it is important to incorporate a nonvanishing β in our discussion. This is easy to do, provided that $\beta = \beta_{\mathbb{R}} \in \mathbb{R}$. We can define $\Theta = \Theta_{\beta_{\mathbb{R}}} + \Theta_{\rho}$ and follow the same steps as in Sections 4 and 5. The result is straightforward but does not cure the problems which appear at higher orders in ρ . The case of generic $\beta \in \mathbb{C}$ is more interesting but also more difficult to study. A noncommutative description of the deformation is not valid in this case and one relies on the $SL(2, \mathbb{R})_s$ symmetry of the supergravity equations of motion in order to construct the dual solution [69]. Consequently, there is no obvious way to incorporate a complex β in our method.

The reason that makes the case of complex β worthwhile to explore further, is that according to the analysis of Section 2, there exist some special points in the deformation space which can take us from a theory of generic β and $\rho = 0$, to a marginal deformation where both $\tilde{\rho}$ and $\tilde{\beta}$ are non-vanishing. Since the gravity dual in the former case is known, investigating the solution at these points may provide useful information on how to extend our results to all orders in the deformation parameters.

There are various possibilities for future work which range from addressing the questions raised above, to establishing a precise connection with generalized complex geometry [83,31], extending the relations between open and closed string parameters to include RR-fields and generalizing the results of [25,86,14] to incorporate supersymmetry. We hope to discuss some of these issues in the future.

7. Note added in proof

Several papers have investigated the subject since this article appeared on the arXiv in Dec. 2006. The dual gravity solution has not been constructed but the relation between exactly marginal deformations and noncommutativity was explored further in [76]. The authors of [76] discussed noncommutativity in the context of quantum groups. In this article, the same noncommutativity matrix was derived from a totally different perspective. Other interesting work on the ρ -deformation includes [46,49,47] in relation to generalized complex geometry and [15,70,75,82,81] regarding integrability and finiteness (see also [7] for some recent developments).

Acknowledgements

I am indebted to A. Murugan and K. Zoubos for numerous discussions that greatly contributed to my understanding of several aspects of this work. I am also grateful to J. Maldacena and L. Rastelli for helpful conversations and encouragement as well as to J.P. Gauntlett, A. Naqvi, A. Parnachev, R. Ricci and M. Roček.

Appendix A. The noncommutativity matrix

Here we present the noncommutativity matrix in polar coordinates (r_i, φ_i) with $i = 1, 2, 3$ on \mathbb{R}^6 . We assume that Θ_{ρ} is given in terms of commuting variables (z, \bar{z}) and that we can follow the transformation rules of contravariant tensors when changing coordinate systems, namely:

$$\Theta^{i'j'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} \Theta^{ij} \quad (\text{A.1})$$

Rescaling q_i of (26) and (27) as $q_i \rightarrow 2q_i$ then yields:

$$\Theta_\rho = \begin{pmatrix} 0 & -(q_2x_3 - q_1y)r_3 & (q_2x_2 - q_1y)r_2 & 0 & \frac{(q_1x_2 - q_2y_1)r_3}{r_2} & \frac{(q_1x_3 - q_2y_1)r_2}{r_3} \\ (q_2x_3 - q_1y)r_3 & 0 & -(q_2x_1 - q_1y)r_1 & -\frac{(q_1x_1 - q_2y_2)r_3}{r_1} & 0 & \frac{(q_1x_3 - q_2y_2)r_1}{r_3} \\ -(q_2x_2 - q_1y)r_2 & (q_2x_1 - q_1y)r_2 & 0 & \frac{(q_1x_1 - q_2y_3)r_2}{r_1} & -\frac{(q_1x_2 - q_2y_3)r_1}{r_2} & 0 \\ 0 & \frac{(q_1x_1 - q_2y_2)r_3}{r_1} & -\frac{(q_1x_1 - q_2y_3)r_2}{r_1} & 0 & -\frac{(q_2x - q_1y_3)r_3}{r_1r_2} & \frac{(q_2x - q_1y_2)r_2}{r_1r_3} \\ -\frac{(q_1x_2 - q_2y_1)r_3}{r_2} & 0 & \frac{(q_1x_2 - q_2y_3)r_1}{r_2} & \frac{(q_2x - q_1y_3)r_3}{r_1r_2} & 0 & -\frac{(q_2x - q_1y_1)r_1}{r_2r_3} \\ \frac{(q_1x_3 - q_2y_1)r_2}{r_3} & -\frac{(q_1x_3 - q_2y_2)r_1}{r_3} & 0 & -\frac{(q_2x - q_1y_2)r_2}{r_1r_3} & \frac{(q_2x - q_1y_1)r_1}{r_2r_3} & 0 \end{pmatrix} \quad (\text{A.2})$$

where to keep the expressions compact, we defined variables x, x_i and y, y_i according to:

$$\begin{aligned} x_1 &= -C_1r_1 + C_2r_2 + C_3r_3, & x_2 &= C_1r_1 - C_2r_2 + C_3r_3, \\ x_3 &= C_1r_1 + C_2r_2 - C_3r_3 \\ y_1 &= -S_1r_1 + S_2r_2 + S_3r_3, & y_2 &= S_1r_1 - S_2r_2 + S_3r_3, & y_3 &= S_1r_1 + S_2r_2 - S_3r_3 \\ x &= C_1r_1 + C_2r_2 + C_3r_3, & y &= S_1r_1 + S_2r_2 + S_3r_3, \end{aligned} \quad (\text{A.3})$$

whereas S_i, C_i represent the following trigonometric functions:

$$\begin{aligned} S_1 &= \sin(\varphi_2 + \varphi_3 - 2\varphi_1), & S_2 &= \sin(\varphi_3 + \varphi_1 - 2\varphi_2), & S_3 &= \sin(\varphi_1 + \varphi_2 - 2\varphi_3) \\ C_1 &= \cos(\varphi_2 + \varphi_3 - 2\varphi_1), & C_2 &= \cos(\varphi_3 + \varphi_1 - 2\varphi_2), & C_3 &= \cos(\varphi_1 + \varphi_2 - 2\varphi_3) \end{aligned} \quad (\text{A.4})$$

The discrete symmetry $\mathbb{Z}_{3(1)} \times \mathbb{Z}_{3(2)}$ along with the $U(1)_R$ are particularly transparent in this form.

Observe first that under $\mathbb{Z}_{3(1)}$:

$$\mathbb{Z}_{3(1)}: (x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (x_3, x_1, x_2, y_3, y_1, y_2) \quad \text{while } (x, y) \rightarrow (x, y) \quad (\text{A.5})$$

Then it is easy to see for example, that $\Theta_\rho^{r_1\varphi_2} = \frac{(q_1x_2 - q_2y_1)r_3}{r_2} \rightarrow \Theta_\rho^{r_3\varphi_1} = \frac{(q_1x_1 - q_2y_3)r_2}{r_1}$.

The action of $\mathbb{Z}_{3(2)}$ is equally simple transforming the polar angles φ_i as:

$$\mathbb{Z}_{3(2)}: (\varphi_1, \varphi_2, \varphi_3) \rightarrow \left(\varphi_1, \varphi_2 + \frac{2\pi}{3}, \varphi_3 - \frac{2\pi}{3} \right) \quad (\text{A.6})$$

thus leaving invariant the trigonometric functions S_i, C_i which depend on the following combinations: $\sigma_i \equiv \frac{1}{3}(\varphi_{i+1} + \varphi_{i+2} - 2\varphi_i)$. Moreover, note that Θ_ρ is independent of $\psi = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3)$ therefore respects the $U(1)_R$ R-symmetry of the theory.

In a similar manner, one obtains the noncommutativity matrix Θ_ρ in spherical coordinates denoted as $(r, \alpha, \theta, \varphi_1, \varphi_2, \varphi_3)$.

$$\begin{aligned} z_1 &= r \cos \alpha e^{i\varphi_1}, & z_2 &= r \sin \alpha \sin \theta e^{i\varphi_2}, & z_3 &= r \sin \alpha \cos \theta e^{i\varphi_3} \\ \bar{z}_1 &= r \cos \alpha e^{-i\varphi_1}, & \bar{z}_2 &= r \sin \alpha \sin \theta e^{-i\varphi_2}, & \bar{z}_3 &= r \sin \alpha \cos \theta e^{-i\varphi_3} \end{aligned} \quad (\text{A.7})$$

where it reads¹³:

¹³ We use here the following abbreviations: $s_\alpha = \sin \alpha, c_\alpha = \cos \alpha, s_\theta = \sin \theta, c_\theta = \cos \theta$.

$$\Theta_\rho = \begin{pmatrix} 0 & -\frac{q_2 u + q_1 v}{s_\alpha} & 0 & \frac{c_\theta(-q_1 u_2 + q_2 v_2)}{s_\alpha s_\theta} & \frac{s_\theta(q_1 u_3 - q_2 v_3)}{s_\alpha c_\theta} \\ \frac{q_2 u + q_1 v}{s_\alpha} & 0 & -\frac{q_1 u_1 + q_2 v_1}{c_\alpha} & \frac{c_\alpha(q_1 u_2 - q_2 v_2)}{s_\alpha^2} & \frac{c_\alpha(q_1 u_3 - q_2 v_3)}{s_\alpha^2} \\ 0 & \frac{q_1 u_1 - q_2 v_1}{c_\alpha} & 0 & \frac{c_\theta(q_2 u_3 + q_1 v_3)}{c_\alpha c_\theta} & -\frac{s_\theta(q_2 u_2 + q_1 v_2)}{s_\alpha^2} \\ \frac{c_\theta(q_1 u_2 - q_2 v_2)}{s_\alpha s_\theta} & \frac{c_\alpha(-q_1 u_2 + q_2 v_2)}{s_\alpha^2} & -\frac{c_\theta(q_2 u_3 + q_1 v_3)}{c_\alpha c_\theta} & 0 & \frac{c_\alpha(q_2 u_1 + q_1 v_1)}{s_\alpha^2 c_\theta} \\ -\frac{s_\theta(q_1 u_3 - q_2 v_3)}{s_\alpha c_\theta} & \frac{c_\alpha(-q_1 u_3 + q_2 v_3)}{s_\alpha^2} & \frac{s_\theta(q_2 u_2 + q_1 v_2)}{c_\alpha c_\theta} & -\frac{c_\alpha(q_2 u_1 + q_1 v_1)}{s_\alpha^2 s_\theta c_\theta} & 0 \end{pmatrix} \quad (\text{A.8})$$

Note that Θ_ρ is now a five-dimensional matrix along the S^5 and that variables u, u_i, v, v_i appearing in (A.8) are defined as:

$$\begin{aligned} u_1 &= (-c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3), & v_1 &= (-c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \\ u_2 &= (c_\alpha C_1 - s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3), & v_2 &= (c_\alpha S_1 - s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3) \\ u_3 &= (c_\alpha C_1 + s_\alpha s_\theta C_2 - s_\alpha c_\theta C_3), & v_3 &= (c_\alpha S_1 + s_\alpha s_\theta S_2 - s_\alpha c_\theta S_3) \\ v &= (c_\alpha S_1 + s_\alpha s_\theta S_2 + s_\alpha c_\theta S_3), & u &= (c_\alpha C_1 + s_\alpha s_\theta C_2 + s_\alpha c_\theta C_3) \end{aligned} \quad (\text{A.9})$$

It is then clear that Θ_ρ is independent of the radial direction r .

Appendix B. RR-fields and supergravity equations of motion

As mentioned previously, although the procedure proposed in this article gives us the solution for the NS–NS fields of the geometry for free, it does not produce any information on the RR-ones. We thus have to compute them using the supergravity equations of motions [97]. We employ the following ansatz¹⁴:

$$\begin{aligned} ds_{10}^2 &= ds_{\text{AdS}_5}^2 + ds_5^2 \\ C &= 0 \quad F_5 = f(\omega_{\text{AdS}_5} + \omega_{\tilde{S}^5}) \end{aligned} \quad (\text{B.1})$$

where f is the appropriate normalization coefficient for the flux which in this case reduces to $f = 16\pi N$ and $\omega_{\text{AdS}_5}, \omega_{\tilde{S}^5}$ are the volume elements of the corresponding parts of the $\text{AdS}_5 \times \tilde{S}^5$ geometry. Then the supergravity field equations reduce to:

$$\begin{aligned} D^2 e^{-2\Phi} &= -\frac{1}{6}(F_3^2 - e^{-2\Phi} H_3^2) \\ F_3 &= -f^{-1} d \star_5 e^{-2\Phi} H_3 \\ H_3 &= f^{-1} d \star_5 F_3 \\ R_{MN} &= -2D_M D_N \Phi - \frac{1}{4} g_{MN} D^2 \Phi + \frac{1}{2} g_{MN} \partial_R \Phi \partial^R \Phi + \frac{1}{96} e^{2\Phi} F_{MPQR} F_N^{PQR} \\ &\quad + \frac{1}{4} (H_{MPQ} H_N^{PQ} + e^{2\Phi} F_{MPQ} F_N^{PQ}) - \frac{1}{48} g_{MN} (H_3^2 + e^{2\Phi} F_3^2) \end{aligned} \quad (\text{B.2})$$

where M, N represent five dimensional indices on the compact piece of the geometry whereas \star_5 denotes the Hodge star on the same manifold.

¹⁴ Note that the vanishing axion condition can be deduced from the other two in Eq. (B.1).

References

- [1] A. Abouelsaood, C.G. Callan, C.R. Nappi, S.A. Yost, Open strings in background gauge fields, *Nucl. Phys. B* 280 (1987) 599.
- [2] O. Aharony, B. Kol, S. Yankielowicz, On exactly marginal deformations of $\mathcal{N} = 4$ SYM and Type IIB supergravity on $\text{AdS}_5 \times S^5$, *J. High Energy Phys.* 0206 (2002) 039.
- [3] O. Aharony, S.S. Razamat, Exactly marginal deformations of $\mathcal{N} = 4$ SYM and of its supersymmetric orbifold descendants, *J. High Energy Phys.* 0205 (2002) 029.
- [4] C. Ahn, J.F. Vazquez-Poritz, Marginal deformations with $U(1)^3$ global symmetry, *J. High Energy Phys.* 0507 (2005) 032.
- [5] C. Ahn, J.F. Vazquez-Poritz, From marginal deformations to confinement, *J. High Energy Phys.* 0606 (2006) 061.
- [6] L.F. Alday, G. Arutyunov, S. Frolov, Green–Schwarz strings in TsT-transformed backgrounds, *J. High Energy Phys.* 0606 (2006) 018.
- [7] A.A. Ardehali, L.A. Pando Zayas, On exactly marginal deformations dual to B -field moduli of IIB theory on SE_5 , arXiv:1405.5290, 2014.
- [8] V. Balasubramanian, R. Gopakumar, F. Larsen, Gauge theory, geometry and the large N limit, *Nucl. Phys. B* 526 (1998) 415–431.
- [9] N. Beisert, R. Roiban, Beauty and the twist: the Bethe ansatz for twisted $\mathcal{N} = 4$ SYM, *J. High Energy Phys.* 0508 (2005) 039.
- [10] D. Berenstein, S.A. Cherkis, Deformations of $N = 4$ SYM and integrable spin chain model, *Nucl. Phys. B* 702 (2004) 49–85.
- [11] D. Berenstein, V. Jejjala, R.G. Leigh, Marginal and relevant deformations of $N = 4$ field theories and non-commutative moduli spaces of vacua, *Nucl. Phys. B* 589 (2000) 196.
- [12] D. Berenstein, R.G. Leigh, Discrete torsion, AdS/CFT and duality, *J. High Energy Phys.* 0001 (2000) 038.
- [13] E. Bergshoeff, C.M. Hull, T. Ortin, Duality in the type-II superstring effective action, *Nucl. Phys. B* 451 (1995) 547–578.
- [14] N. Berkovits, N. Seiberg, Superstrings in graviphoton background and $\mathcal{N} = 1/2 + 3/2$ supersymmetry, *J. High Energy Phys.* 0307 (2003) 010.
- [15] L. Bork, D. Kazakov, G. Vartanov, A. Zhiboedov, Conformal invariance in the Leigh–Strassler deformed $N = 4$ SYM theory, *J. High Energy Phys.* 0804 (2008) 003.
- [16] D. Bundzik, Star product and the general Leigh–Strassler deformation, *J. High Energy Phys.* 0704 (2007) 035.
- [17] D. Bundzik, T. Mansson, The general Leigh–Strassler deformation and integrability, *J. High Energy Phys.* 0601 (2006) 116.
- [18] C.G. Callan, C. Lovelace, C.R. Nappi, S.A. Yost, String loop corrections to beta functions, *Nucl. Phys. B* 288 (1987) 525.
- [19] A. Catal-Ozer, Lunin–Maldacena deformations with three parameters, *J. High Energy Phys.* 0602 (2006) 026.
- [20] A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, *Commun. Math. Phys.* 212 (2000) 591.
- [21] H.-Y. Chen, S.P. Kumar, Precision test of AdS/CFT in Lunin–Maldacena background, *J. High Energy Phys.* 0603 (2006) 051.
- [22] I. Chepelev, A.A. Tseytlin, Long-distance interactions of branes: correspondence between supergravity and super Yang–Mills descriptions, *Nucl. Phys. B* 515 (1998) 73–113.
- [23] L. Cornalba, R. Schiappa, Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds, *Commun. Math. Phys.* 225 (2002) 33.
- [24] M. Cvetič, H. Lu, C.N. Pope, K.S. Stelle, T-duality in the Green–Schwarz formalism and the massless/massive IIA map, *Nucl. Phys. B* 573 (2000) 149–176.
- [25] J. de Boer, P.A. Grassi, P. van Nieuwenhuizen, Non-commutative superspace from string theory, *Phys. Lett. B* 574 (2003) 98–104.
- [26] N. Dorey, A new deconstruction of little string theory, *J. High Energy Phys.* 0407 (2004) 016.
- [27] N. Dorey, S-duality, deconstruction and confinement for a marginal deformation of $\mathcal{N} = 4$ SUSY Yang–Mills, *J. High Energy Phys.* 0408 (2004) 043.
- [28] N. Dorey, T.J. Hollowood, S.P. Kumar, S-duality of the Leigh–Strassler deformation via matrix models, *J. High Energy Phys.* 0212 (2002) 003.
- [29] C. Durnford, G. Georgiou, V. Khoze, Instanton test of non-supersymmetric deformations of the $\text{AdS}_5 \times S^5$, *J. High Energy Phys.* 0609 (2006) 005.

- [30] I. Ellwood, A. Hashimoto, Effective descriptions of branes on non-geometric tori, *J. High Energy Phys.* 0612 (2006) 025.
- [31] I.T. Ellwood, NS–NS fluxes in Hitchin’s generalized geometry, *J. High Energy Phys.* 0712 (2007) 084.
- [32] F. Elmetti, A. Mauri, S. Penati, A. Santambrogio, Conformal invariance of the planar beta-deformed $\mathcal{N} = 4$ SYM theory requires beta real, *J. High Energy Phys.* 0701 (2007) 026.
- [33] D.Z. Freedman, U. Gursoy, Comments on the β -deformed $\mathcal{N} = 4$ SYM theory, *J. High Energy Phys.* 0511 (2005) 042.
- [34] S. Frolov, Lax pair for strings in Lunin–Maldacena background, *J. High Energy Phys.* 0505 (2005) 069.
- [35] S.A. Frolov, R. Roiban, A. Tseytlin, Gauge–string duality for (non)supersymmetric deformations of $\mathcal{N} = 4$ super Yang–Mills theory, *Nucl. Phys. B* 731 (2005) 1–44.
- [36] S.A. Frolov, R. Roiban, A. Tseytlin, Gauge–string duality for superconformal deformations of $\mathcal{N} = 4$ super Yang–Mills theory, *J. High Energy Phys.* 0507 (2005) 045.
- [37] M. Fukuma, T. Oota, H. Tanaka, Comments on T-dualities of Ramond–Ramond potentials, *Prog. Theor. Phys.* 103 (2000) 425–446.
- [38] J.P. Gauntlett, D. Martelli, J. Sparks, Supersymmetric AdS_5 solutions of M-theory, *Class. Quantum Gravity* 21 (2004) 4335–4366.
- [39] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Supersymmetric AdS backgrounds in string and M-theory, in: *Proceedings of Conference, Eur. Math. Soc., 2005*, pp. 217–252, arXiv:hep-th/0411194.
- [40] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Supersymmetric AdS_5 solutions of type IIB supergravity, *Class. Quantum Gravity* 23 (2006) 4693–4718.
- [41] J.P. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, Obstructions to the existence of Sasaki–Einstein metrics, *Commun. Math. Phys.* 273 (2007) 803–827.
- [42] G. Georgiou, V. Khoze, Instanton calculations in the beta-deformed AdS/CFT correspondence, *J. High Energy Phys.* 0604 (2006) 049.
- [43] M.T. Grisaru, B. Milewski, D. Zanon, The structure of UV divergences in SS YM theories, *Phys. Lett. B* 155 (1985) 357.
- [44] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998) 105–114.
- [45] U. Gursoy, C. Nunez, Dipole deformations of $\mathcal{N} = 1$ SYM and supergravity backgrounds with $U(1) \times U(1)$ global symmetry, *Nucl. Phys. B* 725 (2005) 45–92.
- [46] N. Halmagyi, Non-geometric string backgrounds and worldsheet algebras, *J. High Energy Phys.* 0807 (2008) 137.
- [47] N. Halmagyi, Non-geometric backgrounds and the first order string sigma model, arXiv:0906.2891, 2009.
- [48] N. Halmagyi, K. Pilch, C. Romelsberger, N.P. Warner, Holographic duals of a family of $\mathcal{N} = 1$ fixed points, *J. High Energy Phys.* 0608 (2006) 083.
- [49] N. Halmagyi, A. Tomasiello, Generalized Kaehler potentials from supergravity, *Commun. Math. Phys.* 291 (2009) 1–30.
- [50] S.F. Hassan, T-duality, space–time spinors and R–R fields in curved backgrounds, *Nucl. Phys. B* 568 (2000) 145–161.
- [51] M. Herbst, A. Kling, M. Kreuzer, Star products from open strings in curved backgrounds, *J. High Energy Phys.* 0109 (2001) 014.
- [52] M. Herbst, A. Kling, M. Kreuzer, Cyclicity of non-associative products on D-branes, *J. High Energy Phys.* 0403 (2004) 003.
- [53] R. Hernandez, K. Sfetsos, D. Zoakos, Gravity duals for the Coulomb branch of marginally deformed $\mathcal{N} = 4$ Yang–Mills, *J. High Energy Phys.* 0603 (2006) 069.
- [54] P.-M. Ho, Making non-associative algebra associative, *J. High Energy Phys.* 0111 (2001) 026.
- [55] D.R.T. Jones, Coupling constant reparameterization and finite field theories, *Nucl. Phys. B* 277 (1986) 153.
- [56] D.R.T. Jones, L. Mezincescu, The chiral anomaly and a class of two-loop finite supersymmetric gauge theories, *Phys. Lett. B* 138 (1984) 293.
- [57] S. Kachru, E. Silverstein, 4d conformal field theories and strings on orbifolds, *Phys. Rev. Lett.* 80 (1998) 4855–4858.
- [58] V.V. Khoze, Amplitudes in the beta-deformed conformal Yang–Mills, *J. High Energy Phys.* 0602 (2006) 040.
- [59] I. Klebanov, E. Witten, Superconformal field theory on threebranes at a Calabi–Yau singularity, *Nucl. Phys. B* 536 (1998) 199–218.
- [60] I.R. Klebanov, M. Strassler, Supergravity and a confining gauge theory: duality cascades and χ SB-resolution of naked singularities, *J. High Energy Phys.* 0008 (2000) 052.
- [61] M. Kontsevich, Deformation quantization of Poisson manifolds I, *Lett. Math. Phys.* 66 (2003) 157.

- [62] M. Kulaxizi, On beta-deformations and noncommutativity, arXiv:hep-th/0610310, 2006.
- [63] M. Kulaxizi, K. Zoubos, Marginal deformations of tree-level $\mathcal{N} = 4$ SYM from twistor string theory, *Fortschr. Phys.* 54 (2006) 423–429.
- [64] M. Kulaxizi, K. Zoubos, Marginal deformations of $\mathcal{N} = 4$ SYM from open/closed twistor strings, *Nucl. Phys. B* 738 (2006) 317–349.
- [65] S.M. Kuzenko, A. Tseytlin, Effective action of β -deformed $\mathcal{N} = 4$ SYM theory and AdS/CFT, *Phys. Rev. D* 72 (2005) 075005.
- [66] A. Lawrence, N. Nekrasov, C. Vafa, On conformal theories in four dimensions, *Nucl. Phys. B* 533 (1998) 199–209.
- [67] R.G. Leigh, M.J. Strassler, Exactly marginal operators and duality in four dimensional $\mathcal{N} = 1$ supersymmetric gauge theory, *Nucl. Phys. B* 447 (1995) 95.
- [68] D. Lowe, H. Nastase, S. Ramgoolam, Massive IIA string theory and matrix theory compactification, *Nucl. Phys. B* 667 (2003) 55–89.
- [69] O. Lunin, J. Maldacena, Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals, *J. High Energy Phys.* 0505 (2005) 033.
- [70] K. Madhu, S. Govindarajan, Chiral primaries in the Leigh–Strassler deformed $N = 4$ SYM: a perturbative study, *J. High Energy Phys.* 0705 (2007) 038.
- [71] J. Maldacena, Branes probing black holes, *Nucl. Phys. B, Proc. Suppl.* 68 (1998) 17–27.
- [72] J. Maldacena, The large N limit of superconformal theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998) 231.
- [73] J. Maldacena, Probing near extremal black holes with D-branes, *Phys. Rev. D* 57 (1998) 3736–3741.
- [74] J. Maldacena, C. Nunez, Towards the large N limit of pure $\mathcal{N} = 1$ super Yang Mills, *Phys. Rev. Lett.* 86 (2001) 588–591.
- [75] T. Mansson, The Leigh–Strassler deformation and the quest for integrability, *J. High Energy Phys.* 0706 (2007) 010.
- [76] T. Mansson, K. Zoubos, Quantum symmetries and marginal deformations, *J. High Energy Phys.* 1010 (2010) 043.
- [77] A. Mauri, S. Penati, M. Pirrone, A. Santambrogio, D. Zanon, On the perturbative chiral ring for marginally deformed $\mathcal{N} = 4$ SYM theories, *J. High Energy Phys.* 0608 (2006) 072.
- [78] A. Mauri, S. Penati, A. Santambrogio, D. Zanon, Exact results in planar $\mathcal{N} = 1$ superconformal Yang–Mills theory, *J. High Energy Phys.* 0511 (2005) 024.
- [79] T. McLoughlin, I. Swanson, Integrable twists in AdS/CFT, *J. High Energy Phys.* 0608 (2006) 084.
- [80] R.R. Metsaev, A.A. Tseytlin, Supersymmetric D3 brane actions in $AdS_5 \times S^5$, *Phys. Lett. B* 436 (1998) 281–288.
- [81] J. Minahan, C. Sieg, Four-loop anomalous dimensions in Leigh–Strassler deformations, *J. Phys. A* 45 (2012) 305401.
- [82] J.A. Minahan, Supergraphs and the cubic Leigh–Strassler model, *J. High Energy Phys.* 1112 (2011) 093.
- [83] R. Minasian, M. Petrini, A. Zaffaroni, Gravity duals to deformed SYM theories and generalized complex geometry, *J. High Energy Phys.* 0612 (2006) 055.
- [84] V. Niarchos, N. Prezas, BMN operators for $N = 1$ superconformal Yang–Mills theories and associated string backgrounds, *J. High Energy Phys.* 06 (2003) 015.
- [85] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Beta function in supersymmetric gauge theories: instantons versus traditional approach, *Phys. Lett. B* 166 (1986) 329.
- [86] H. Ooguri, C. Vafa, The C-deformation of gluino and non-planar diagrams, *Adv. Theor. Math. Phys.* 7 (2003) 53–85.
- [87] A. Parkes, P.C. West, Finiteness in rigid supersymmetric theories, *Phys. Lett. B* 138 (1984) 99.
- [88] A.J. Parkes, P.C. West, Three-loop results in two-loop finite supersymmetric gauge theories, *Nucl. Phys. B* 256 (1985) 340.
- [89] S. Penati, A. Santambrogio, D. Zanon, Two point correlators in the beta-deformed $\mathcal{N} = 4$ SYM at the next-to-leading order, *J. High Energy Phys.* 0510 (2005) 023.
- [90] K. Pilch, N.P. Warner, A new supersymmetric compactification of chiral IIB supergravity, *Phys. Lett. B* 487 (2000) 22–29.
- [91] K. Pilch, N.P. Warner, $N = 2$ supersymmetric RG flows and the IIB dilaton, *Nucl. Phys. B* 594 (2001) 209–228.
- [92] R.C. Rashkov, K.S. Viswanathan, Y. Yang, Generalizations of Lunin–Maldacena transformation on the $AdS_5 \times S^5$ background, *Phys. Rev. D* 72 (2005) 106008.
- [93] O. Aharony, S.S. Razamat, Marginal deformations of $N = 4$ SYM and of its supersymmetric orbifold descendants, *J. High Energy Phys.* 0205 (2002) 029.
- [94] G. Rossi, E. Sokatchev, Y. Stanev, New results in the deformed $N = 4$ SYM theory, *Nucl. Phys. B* 729 (2005) 581–593.

- [95] G. Rossi, E. Sokatchev, Y. Stanev, On the all-order perturbative finiteness of the deformed $N = 4$ SYM theory, Nucl. Phys. B 754 (2006) 329–350.
- [96] V. Schomerus, D-branes and deformation quantization, J. High Energy Phys. 9906 (1999) 030.
- [97] J.H. Schwarz, Covariant field equations of chiral $N = 2$ $D = 10$ supergravity, Nucl. Phys. B 226 (1983) 269.
- [98] N. Seiberg, E. Witten, String theory and noncommutative geometry, J. High Energy Phys. 9909 (1999) 032.
- [99] J. Shelton, W. Taylor, B. Wecht, Nongeometric flux compactifications, J. High Energy Phys. 0510 (2005) 085.
- [100] M.A. Shifman, A.I. Vainshtein, Solution of the anomaly puzzle in SUSY gauge theories and the Wilson operator expansion, Nucl. Phys. B 277 (1986) 456.
- [101] M.A. Shifman, A.I. Vainshtein, On holomorphic dependence and infrared effects in supersymmetric gauge theories, Nucl. Phys. B 359 (1991) 571.
- [102] K. Skenderis, M. Taylor, Holographic Coulomb branch vevs, J. High Energy Phys. 0608 (2006) 001.
- [103] K. Skenderis, M. Taylor, Kaluza–Klein holography, J. High Energy Phys. 0605 (2006) 057.
- [104] M.F. Sohnius, Introducing supersymmetry, Phys. Rep. 128 (1985) 39.
- [105] P. Sundell, Spin $(p + 1, p + 1)$ covariant D_p -brane bound states, Int. J. Mod. Phys. A 16 (2001) 3025–3040.
- [106] A.A. Tseytlin, Born–Infeld action, supersymmetry and string theory, in: M.A. Shifman (Ed.), The Many Faces of the Superworld, 2000, pp. 417–452, arXiv:hep-th/9908105.
- [107] M. Wijnholt, Parameter space of quiver gauge theories, Adv. Theor. Math. Phys. 12 (2008).
- [108] E. Witten, Anti de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291.